PID Design with Filter for Servo Motor Applications

Abstract:

This document is intended to serve as a design tool for PID design in motor servo systems.
It begins with the PID Velocity Control Loop Model and a DC Motor Model. Then target specifications are defined and by application of a PID Direct Synthesis Method, the Controller is designed. Analysis in both the continuous and discrete frequency domain is presented. A second order, critically damped filter, is detailed to be included in the closed loop and the selection of the sampling frequency is discussed. Time delays are modeled and included in this analysis.

This document is divided into 12 parts:

Part 1 – Introduction and PID with Filter Model
Part 2 – Velocity Loop with Motor Model
Part 3 – PID Direct Synthesis Method
Part 4 - PID Analysis in the Continuous Time and Discrete Time Domains
Part 5 – The Filter in the Continuous Domain and Discrete Domain
Part 6 – Inclusion of Time Delays; Filter and Zero Order Hold (ZOH).
Part 7 – Open and Closed Loop Gain and Phase
Part 8 – Step Response of the Finished Design
Part 9 – Changing The Closed Loop Bandwidth
Part 10 – Integral Anti-windup
Part 11 – Summary
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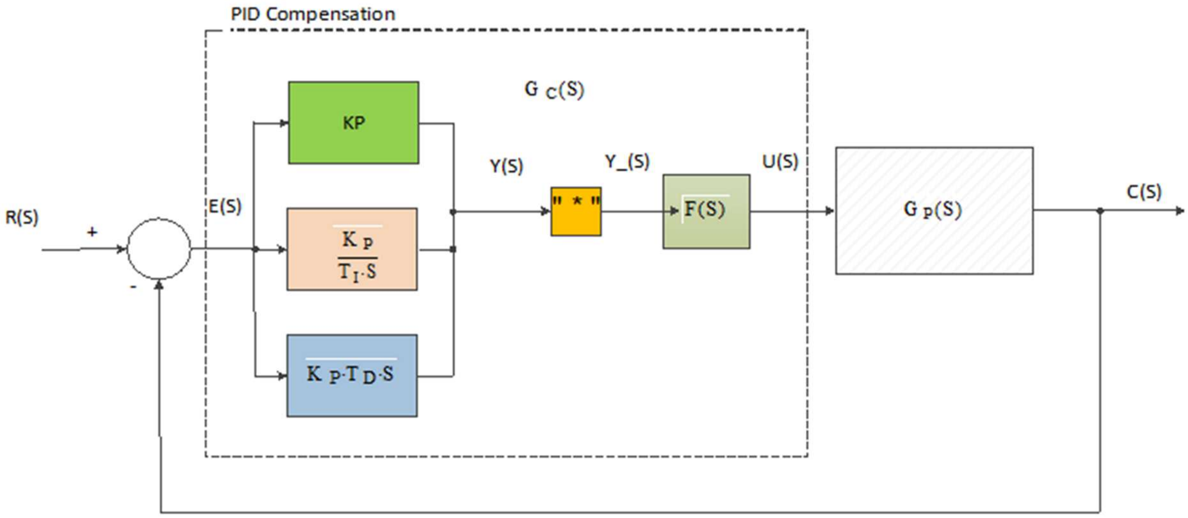
Part 1 – Introduction and PID with Filter Model

This document introduces a filter to be used at the controller output so that the proportional, integral and derivative outputs share the same time delay for all parameters. Refer to Figure 1 below.



$$G_C(S) = \frac{Y(S)}{E(S)} = \frac{Y\_(S)}{E(S)}$$

Velocity Loop Servo Motor Controller using a PID with an output Filter
Figure 1

The Block " * " is a placeholder to be used at the end of this document.
It is treated as a Unity Gain Block for now.

Standard Form for a PID Controller is given by EQ 1

$$G_C(S) = K_P \cdot \left(1 + \frac{1}{S \cdot T_I} + S \cdot T_D\right) \qquad \textbf{EQ 1}$$

Units

K_P $\qquad \dfrac{V}{\frac{rad}{s}} \qquad \dfrac{V}{\frac{rad}{s}} \cdot \dfrac{rad}{s} = 1V$

$K_I = \dfrac{K_p}{T_I} \qquad \dfrac{\frac{V}{\left(\frac{rad}{s}\right)}}{s} = 1\dfrac{V}{rad} \qquad \dfrac{V}{rad} \cdot \dfrac{rad}{s} = 1\dfrac{V}{s}$

$K_D = K_P \cdot T_D \qquad \dfrac{V}{\left(\frac{rad}{s}\right)} \cdot s = 1\dfrac{V}{\frac{rad}{s^2}} \qquad \dfrac{V}{\frac{rad}{s^2}} \cdot \dfrac{rad}{s} = 1V \cdot s$

PID parts

$$G_{CP}(S) = K_P \qquad \textbf{EQ 2}$$

$$G_{CI}(s) = \frac{K_P}{T_I} \cdot \frac{1}{S} \qquad \textbf{EQ 3}$$

$$G_{CD}(S) = K_P \cdot T_D \cdot S \qquad \textbf{EQ 4}$$

$$G_C(S) = G_{CP}(S) + G_{CI}(S) + G_{CD}(S) \qquad \textbf{EQ 5}$$

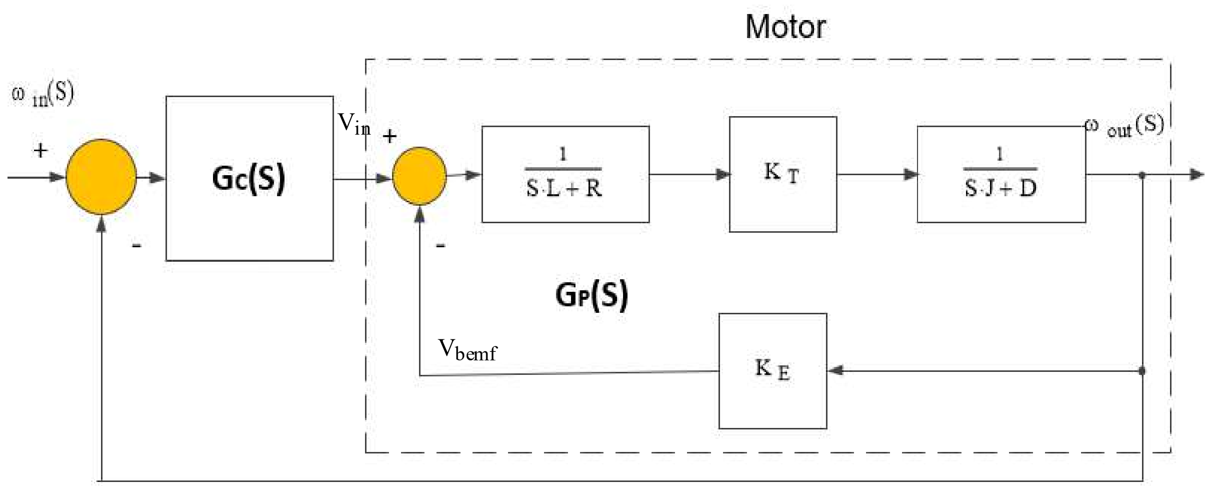
A PID Controller with an Output filter, F(S) as depicted in Figure 1 above is given by EQ 6:

$$G_{CF}(S) = G_C(S) \cdot F(S) \qquad \textbf{EQ 6}$$

Part 2 – Velocity Loop with Motor Model

Before we discuss the control loop design, we will focus on the DC Motor Model. Refer to Figure 2.

DC Motor



Velocity Loop with DC Motor Model
Figure 2.

The Transfer Function of the Motor is given by:

$$\frac{\omega_{out}(S)}{V_{in}(S)} = G_P(S) = \frac{K_T}{S^2\cdot(L\cdot J)+S\cdot(R\cdot J+D\cdot L)+\left(K_T\cdot K_E+R\cdot D\right)} \qquad \frac{\frac{rad}{s}}{V}$$

Second Order Process

$$G_P(S) = \frac{\frac{K_T}{L\cdot J}}{S^2+S\cdot\left(\frac{R}{L}+\frac{D}{J}\right)+\frac{\left(K_T\cdot K_E+R\cdot D\right)}{L\cdot J}} \qquad \frac{\frac{rad}{s}}{V} \qquad \textbf{EQ 7}$$

For an example of an underdamped motor - use Example 2 Motor from *Introduction to Brushless Motor Parameters and Figures of Merit* by Frederick York. **BEI Kimco DIP30-07-001A**

$L := 8 \cdot mH$ $\qquad R := 5.6 \cdot \Omega$

$J := 1.36 \cdot 10^{-5} \cdot kg \cdot m^2$ $\qquad D := 8.69882 \cdot 10^{-5} \cdot \dfrac{N \cdot m}{\frac{rad}{s}}$ $\qquad$ ** **Correction**

$K_T := 0.1957 \cdot \dfrac{N \cdot m}{A}$ $\qquad K_E := 0.226 \cdot \dfrac{V}{\frac{rad}{s}}$ $\qquad$ Assumes a sine wave drive

$\tau_e := \dfrac{L}{R}$ $\qquad \tau_m := \dfrac{J}{D}$

$\tau_e = 0.00143\,s$ $\qquad \tau_m = 0.15634\,s$

$\Omega_n := \sqrt{\dfrac{(K_T \cdot K_E + R \cdot D)}{L \cdot J}}$ $\qquad \Omega_n = 641.082\,\dfrac{rad}{s}$

$\zeta := \dfrac{\frac{1}{2} \cdot \left(\frac{R}{L} + \frac{D}{J}\right)}{\Omega_n}$ $\qquad \zeta = 0.55094$

This is a second order underdamped Process. The method outlined in the document Cascade PI *Controller Design using a Limited Span Method, by Frederick York,* provides one solution.

In the document *Cascade* PI *Controller Design using a Limited Span Method, by Frederick York,* it was listed in the Summary Table on Page 8 that $5\ \omega_D < \omega_p < \omega_{sample\_rate} / 10$. For $\omega_D = 100$ rad/s, and $\omega_{sample\_rate} = 2\ \pi\ 10{,}000$ rad/s the ratio is easily met. Therefore, the Example 2 analysis, where $\chi = 2.716$ for a $\omega_{3dBf} = 22.06$ rad/s is preserved for this motor. Note that the setpoint Lag Filter is still needed $\omega_z(2.716) = 13.556$ rad/s.

It is this example to which the following PID design is compared.

Part 3 – PID Direct Synthesis Method

$$G_C(S) = \frac{\frac{c(S)}{r(S)}}{G_P(S)\cdot\left(1 - \frac{c(S)}{r(S)}\right)}$$

EQ 8 **Ignore F(S) for now**

$r(S)$ is the setpoint $c(S)$ is the desired output $G_P(S)$ is the "practical" or desired respon

Then $G_C(S)$ is the required controller transfer function to make this happen.

For this motor example we include a transport delay, θ (To be addressed later)

$$G_P(S) = \frac{k\cdot \exp(-\theta\cdot S)}{S^2\cdot\tau^2 + S\cdot 2\cdot\zeta\,\tau + 1}$$

EQ 9

and $$\frac{c(S)}{r(S)} = \frac{\exp(-\theta\cdot S)}{1 + \tau_c\cdot S}$$ **EQ 10**

Using EQ 8, 9 and 10 we get:

$$G_C(S) = \frac{\left(S^2\cdot\tau^2 + S\cdot 2\cdot\zeta\,\tau + 1\right)}{k\cdot\exp(-\theta\cdot S)}\cdot\frac{\frac{1}{1+\tau_c\cdot S}\cdot\exp(-\theta\cdot S)}{\left(1 - \frac{\exp(-\theta\cdot S)}{1+\tau_c\cdot S}\right)}$$

$$= \left(\frac{S^2\cdot\tau^2 + S\cdot 2\cdot\zeta\cdot\tau + 1}{k}\right)\cdot\frac{1}{\left[\left(1+\tau_c\cdot S\right) - \exp(-\theta\cdot S)\right]}$$

Approximating $\exp(-\theta\cdot S)$ as $=$ $1 - \theta\cdot S$

$$G_C(S) = \frac{\left(S^2\cdot\tau^2 + S\cdot 2\cdot\zeta\cdot\tau + 1\right)}{k\cdot S\cdot\left(\tau_c + \theta\right)}$$ **EQ 11**

Then

$$G_C(S) = \frac{2\cdot\zeta\,\tau}{k\cdot\left(\tau_c+\theta\right)}\cdot\left(S\cdot\frac{\tau}{2\cdot\zeta} + \frac{1}{2\cdot\zeta\,\tau\cdot S} + 1\right)$$ **EQ 12**

Equating to terms in EQ 1 -

$$K_P = \frac{2\cdot\zeta\,\tau}{k\cdot\left(\tau_c + \theta\right)}$$ **EQ 12a**

$$T_I = 2\cdot\zeta\cdot\tau$$ **EQ 12b**

$$T_D = \frac{\tau}{2\cdot\zeta}$$ **EQ 12c**

$$\tau = 1.55986\,\text{milli}\cdot s \qquad \zeta = 0.55094$$

If we desire a closed loop bandwidth of 22 rad / s --

$$\tau_c := \frac{1}{22\cdot\frac{rad}{s}} \qquad \tau_c = 45.455\times10^{-3}\,s$$

Select $\alpha := 0.1$

and

$$k := \frac{\frac{K_T}{L\cdot J}}{\frac{\left(K_E\cdot K_T + R\cdot D\right)}{L\cdot J}} \qquad k := 4.376575 \quad \left(\frac{\frac{rad}{s}}{V}\right)$$

From EQ 7, S = 0

$$K_P := \frac{2\cdot\zeta\cdot\tau}{k\cdot\left(\tau_c+\theta\right)} \qquad K_P := 0.008599 \quad \left(\frac{V}{\frac{rad}{s}}\right)$$

Assume for now that θ = 215.55 * µs
We will derive this later.

$$T_D := \frac{\tau}{2\cdot\zeta} \qquad T_D = 1415.6362\,\mu\cdot s$$

$$T_I := 2\cdot\zeta\cdot\tau \qquad T_I = 1.7188\,\text{milli}\cdot s$$

$$T_f := \alpha\cdot T_D \qquad T_f = 141.564\,\mu\cdot s \qquad \frac{1}{T_f} = 7063.96\,\frac{rad}{s}$$

$F_x := 1.553774$

This adjusts the double pole frequency location so that the 3 dB frequency, (Continuous Time Frequency Domain) is

+

located at $\frac{1}{T_f}$

$\gamma := \frac{T_f}{F_x}$ $\gamma = 91.11\,\mu\cdot s$ $\frac{1}{\gamma} = 10975.8\,\frac{rad}{s}$

$\frac{1}{\tau_c} = 22\,\frac{rad}{s}$ $\frac{1}{\tau} = 641.082\,\frac{rad}{s}$

Derivation of Fx value

For a Second Order Filter -

$$F(\Omega) = \frac{1}{\left[1-\left(\frac{\Omega}{\Omega_{nF}}\right)^2\right]+\left[2\cdot\zeta_F\cdot\left(\frac{\Omega}{\Omega_{nF}}\right)\right]} \qquad |F(\Omega)| = \frac{1}{\sqrt{\left[1-\left(\frac{\Omega}{\Omega_{nF}}\right)^2\right]^2+\left[2\cdot\zeta_F\cdot\left(\frac{\Omega}{\Omega_{nF}}\right)\right]^2}}$$

At the 3dB frequency Ω_{3dB}

$$\frac{1}{\sqrt{\left[1-\left(\frac{\Omega_{3dB}}{\Omega_{nF}}\right)^2\right]^2+\left[2\cdot\zeta_F\cdot\left(\frac{\Omega_{3dB}}{\Omega_{nF}}\right)\right]^2}} = \frac{1}{\sqrt{2}} \qquad \left[1-\left(\frac{\Omega_{3dB}}{\Omega_{nF}}\right)^2\right]^2+\left[2\cdot\zeta_F\cdot\left(\frac{\Omega_{3dB}}{\Omega_{nF}}\right)\right]^2 = 2$$

Expanding

$$\left(\frac{\Omega_{3dB}}{\Omega_{nF}}\right)^4 - 2\cdot\left(\frac{\Omega_{3dB}}{\Omega_{nf}}\right)^2\cdot\left(2\cdot\zeta_F - 1\right) - 1 = 2$$

Solving for $\frac{\Omega_{nF}}{\Omega_{3dB}} = F_x$

$$F_x = \left[\sqrt{1 - 2\cdot\zeta_F + \sqrt{\left(2\cdot{\zeta_F}^2 - 1\right)^2 + 1}}\right]^{-1}$$

For $\zeta_F = 1$

$F_x = 1.553774$

Part 4 - PID Analysis in the Continuous Time and Discrete Time Domains

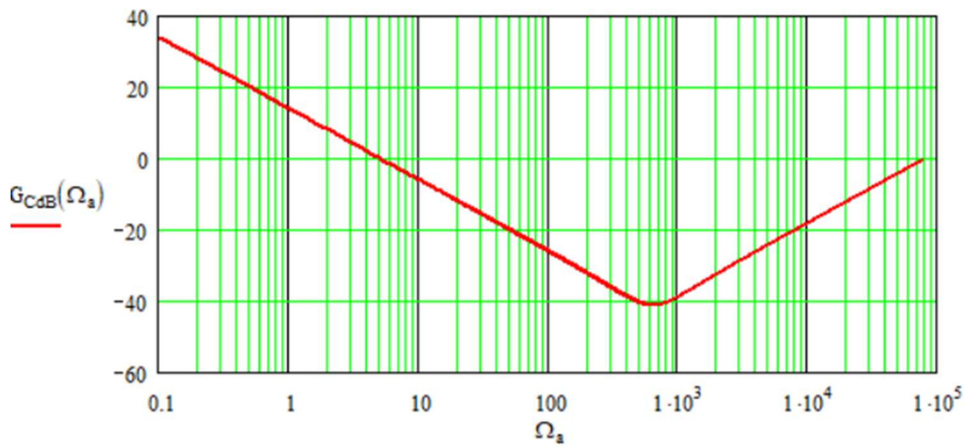
In the Continuous Time Frequency Domain

Substituting $S = j \cdot \Omega_a$ into EQ 1

$$G_C(\Omega_a) := K_P \cdot \left(1 + \frac{1}{j \cdot \Omega_a \cdot T_I} + j \cdot \Omega_a \cdot T_D\right) \qquad \textbf{EQ 13}$$

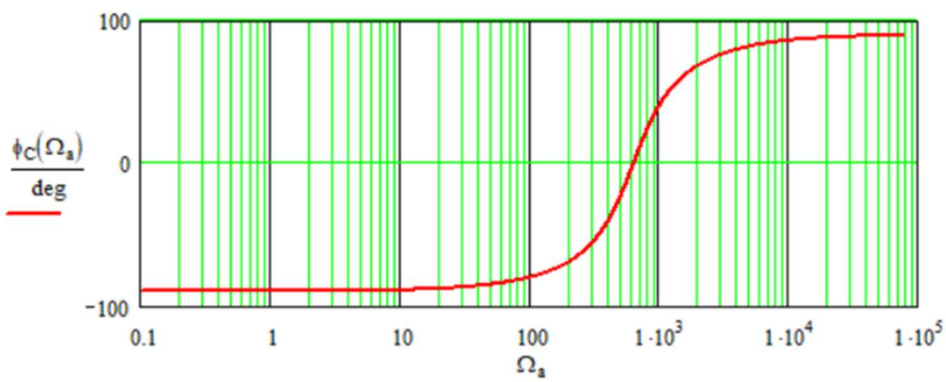
$$G_{CdB}(\Omega_a) := 20 \cdot \log\left(\left|G_C(\Omega_a)\right|\right)$$

$$\phi_C(\Omega_a) := \arg\left(G_C(\Omega_a)\right)$$



PID Gain Plot – Continuous Domain

Figure 3

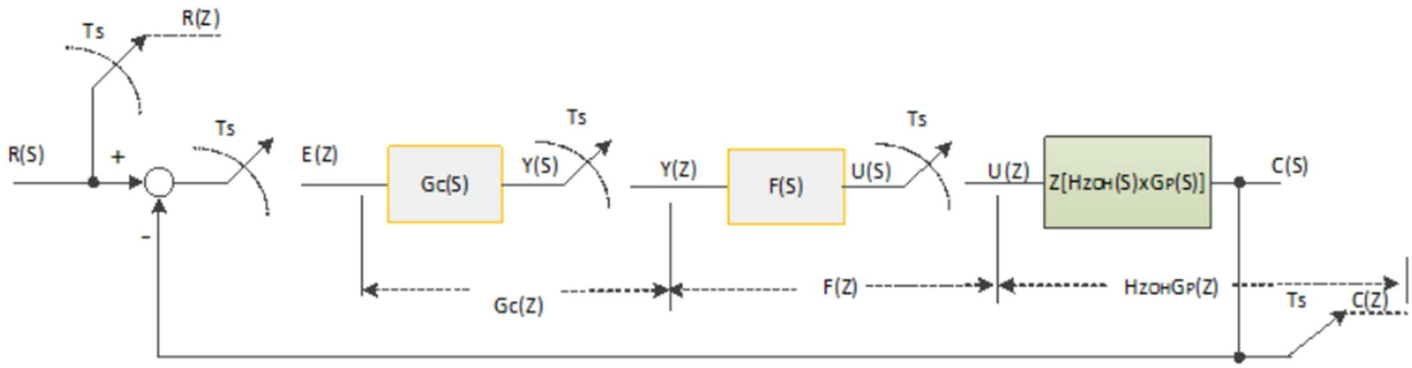


PID Phase Plot – Continuous Domain

Figure 4

PID Implementation - Discrete Time Domain

As the motor is a 3 phase sine / sine drive, it is expected to be implemented on a microcontroller / DSP. There is an implied inverter to accommodate the drive with a sine look-up. Therefore, a discrete solution is presented.



Velocity Loop Servo - Discrete Model

Figure 5

Note that the Zero Order Hold models the analog waveform generation (any number of phases). Some examples are Sine PWM, SVFPWM and Trapezoidal PWM.

Closed Loop Equation

$$G_{CL}(Z) = \frac{C(Z)}{R(Z)} = \frac{G_C(Z)F(Z)H_{ZOH}G_P(Z)}{1 + G_C(Z)F(Z)H_{ZOH}G_P(Z)}$$ **EQ 14**

Definitions

f_s = sample frequency /rate (samples / s)

$f_{Nyqst} := \frac{f_s}{2}$ Nyquist Rate

$T_s := \frac{1}{f_s}$ $T_s = 66.6667 \frac{\mu \cdot s}{sample}$

In this Document ---

$f_s := 15000 \cdot \frac{samples}{s}$

$f_{Nyqst} = 7500 \frac{samples}{s}$

$\omega_{Nyqst} := 2 \cdot \pi \cdot f_{Nyqst}$

We will return to the sample rate (15 KHz) when discussing the Filter section to follow.

$$H_d(Z) = H_a(S) \Big|_{S = \frac{2}{T_s}\cdot\frac{Z-1}{Z+1}} = H_a\left(\frac{2}{T_s}\cdot\frac{Z-1}{Z+1}\right)$$

Relating the two domains via the Bilinear Transformation (Tusin's / Trapaazoidal Method)

$$S = \frac{2}{T_s}\cdot\frac{Z-1}{Z+1} \qquad \textbf{EQ 15}$$

$$H_d\cdot(\exp(j\cdot\omega)) = H_a(j\cdot\Omega_a) = H_a\left(j\cdot\frac{2}{T_s}\cdot\tan\left(\frac{\omega\cdot T_s}{2}\right)\right)$$

Relationships

$$\omega_d := 0\cdot\text{rad}, 0.01\cdot\text{rad} .. \pi\cdot[(1-0.001)\text{rad}]$$

$$\Omega_A(\omega_d) := \frac{2}{T_s}\cdot\tan\left(\frac{\omega_d}{2}\right) \qquad \textbf{EQ 16}$$

Discrete Time Frequency Domain to Continuous Time Frequency Domain

$$\omega_D(\Omega_a) := 2\cdot\text{atan}\left(\frac{\Omega_a\cdot T_s}{2}\right) \qquad \textbf{EQ 17}$$

Continuous Time Frequency Domain to Discrete Time Frequency Domain

$$\omega_{\Omega T}(\Omega_a) := \Omega_a\cdot T_s\left(\frac{\text{rad}}{\text{sample}}\right) \qquad \textbf{EQ 18}$$

Linear Discrete Frequency

$-\infty < \Omega_A \le \infty$ $\quad\left(\frac{\text{rad}}{\text{s}}\right)$ Continuos Time Radian Frequency Range

$-\pi < \omega_d \le \pi$ $\quad(\text{rad})$ Discrete Time Frequency Range

$\frac{-f_s}{2} < f \le \frac{f_s}{2}$ $\quad(\text{Hz})$ **Allowable Frequency Range (No aliasing)**

$$\Omega_A(\omega_d) \sim \frac{\omega_D(\Omega_a)}{T_s}$$

When

$\frac{\Omega_a\cdot T_s}{2}$ and $\frac{\omega_d}{2} \le \varepsilon$

$\varepsilon = 0.17$ for 1% accuracy **Cond. 1**

$\varepsilon = 0.3$ ~ 3.11 % accuracy **Cond. 2**

As compared to $\Omega_a = \varepsilon \cdot \frac{2}{T_s}$

or $\omega_d = 2 \cdot \varepsilon$

$$G_C(S) = K_P \cdot \left(1 + \frac{1}{S \cdot T_I} + S \cdot T_D\right)$$ **"EQ 1"** Continuous Domain, S

Using the Bilinear Transform Formula -

$$S = \frac{2}{T_s} \cdot \frac{1 - Z^{-1}}{1 + Z^{-1}}$$ **"EQ 15"**

to map EQ 1 to the Z domain.

$$G_{C\_Z}(Z) = G_C\left(S = \frac{2}{T_s} \cdot \frac{1 - Z^{-1}}{1 + Z^{-1}}\right) = K_P \cdot \left[1 + \frac{T_s}{2 \cdot T_I} \cdot \frac{\left(1 + Z^{-1}\right)}{\left(1 - Z^{-1}\right)} + \frac{2 \cdot T_D}{T_s} \cdot \frac{\left(1 - Z^{-1}\right)}{\left(1 + Z^{-1}\right)}\right]$$ **EQ 19**

Expanding

$$\frac{1}{2} \cdot \frac{K_P \cdot \left[Z^2 \cdot \left(2 \cdot T_I \cdot T_s + T_s^{\,2} + 4 \cdot T_D \cdot T_I\right) + Z \cdot \left(2 \cdot T_s^{\,2} - 8 \cdot T_D \cdot T_I\right) + \left(4 \cdot T_D \cdot T_I - 2 \cdot T_I \cdot T_s + T_s^{\,2}\right)\right]}{T_I \cdot T_s \cdot \left(Z^2 - 1\right)}$$

Let

$$k_0 := \frac{K_P}{2 \cdot T_I \cdot T_s} \cdot \left(4 \cdot T_D \cdot T_I - 2 \cdot T_I \cdot T_s + T_s^2\right)$$

$$k_1 := \frac{K_P}{2 \cdot T_I \cdot T_s} \cdot \left(2 \cdot T_s^2 - 8 \cdot T_D \cdot T_I\right)$$

$$k_2 := \frac{K_P}{2 \cdot T_I \cdot T_s} \cdot \left(2 \cdot T_I \cdot T_s + T_s^2 + 4 \cdot T_D \cdot T_I\right) \qquad p := 1$$

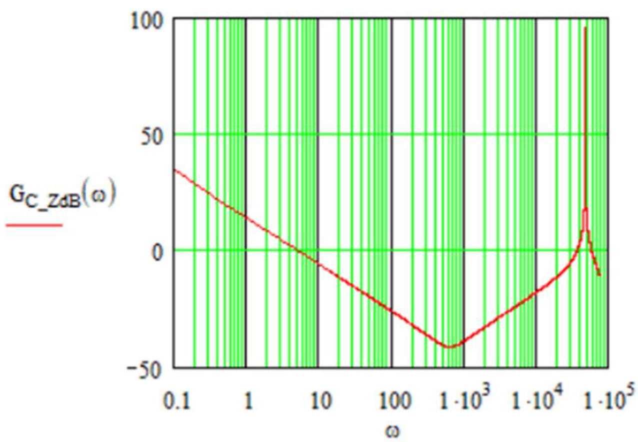
$$G_{C\_Z}(Z) = \frac{k_0 + k_1 \cdot Z^{-1} + k_2 \cdot Z^{-2}}{1 - p \cdot Z^{-2}} \qquad \textbf{EQ 20}$$

In the discrete frequency domain:

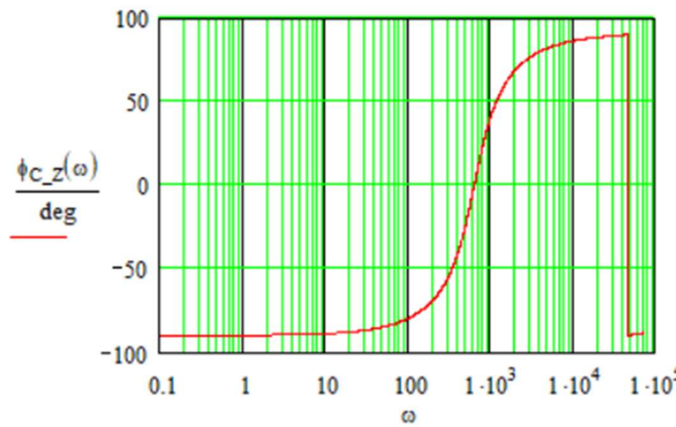
$$Z(\omega) := \exp(j \cdot \omega \cdot T_s)$$

$$G_{C\_Z}(\omega) := \frac{k_2 \cdot Z(\omega)^2 + k_1 \cdot Z(\omega) + k_0}{Z(\omega)^2 - 1} \qquad \textbf{EQ 21}$$

$$G_{C\_ZdB}(\omega) := 20 \cdot \log\left(\left|G_{C\_Z}(\omega)\right|\right) \qquad \phi_{C\_Z}(\omega) := \arg\left(G_{C\_Z}(\omega)\right)$$



Discrete Amplitude Response
Figure 6



Discrete Phase Response
Figure 7

Part 5 – The Filter in the Continuous Domain and Discrete Domain

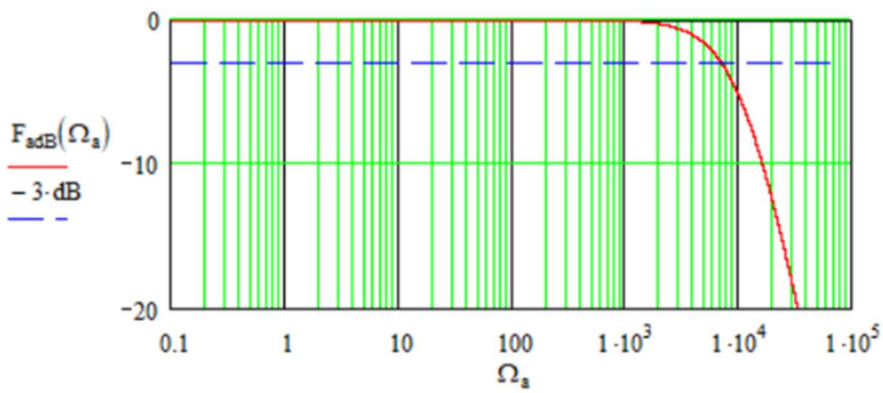
<u>Continuous Time Filter in the frequency Domain</u>

$$F(S) = \frac{1}{(1+S\cdot\gamma)^2}$$ **EQ 22** $$\omega_{nF} := \frac{1}{\gamma}$$ $$\zeta_F := 1$$

$$S = j\cdot\Omega_a$$

$$F_a(\Omega_a) := \frac{1}{\left[1+j\cdot(\Omega_a\cdot\gamma)\right]^2}$$ **EQ 23**

$$F_{adB}(\Omega_a) := 20\cdot\log\left(\left|F_a(\Omega_a)\right|\right) \qquad \phi_{aF}(\Omega_a) := \arg\left(F_a(\Omega_a)\right)$$



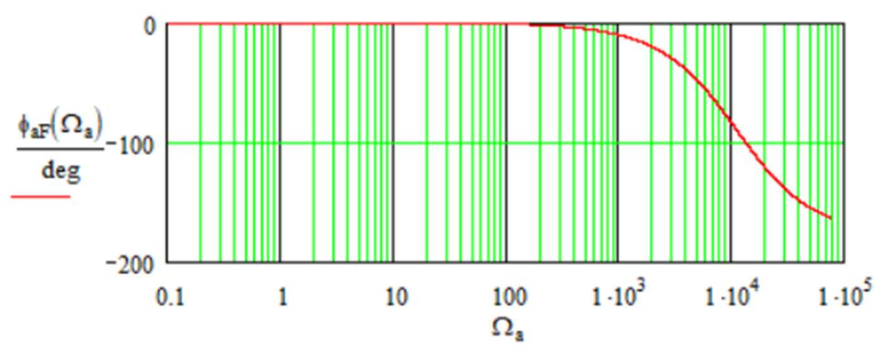
Actual 3 dB frequency

$$F_{adB}\left(\frac{1}{\gamma\cdot F_x}\right) = -3.01\,dB$$

$$W_{f3dB} := 7063.962\cdot\frac{rad}{s}$$

Continuous Time Amplitude Response

Figure 8

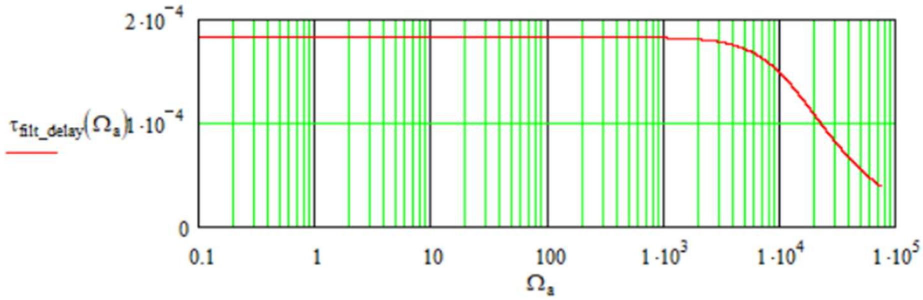


$$\phi_{aF}\left(\frac{1}{\gamma\cdot F_x}\right) = -65.53\,deg$$

Continuous Time Phase Response

Figure 9

$$\tau_{filt\_delay}(\Omega_a) := \frac{-\phi_{aF}(\Omega_a)}{\Omega_a}$$



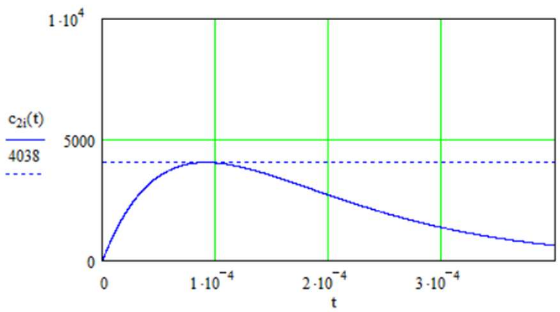
$$\tau_{filt\_delay}\left(1 \cdot \frac{rad}{s}\right) = 182.22\,\mu \cdot s$$

Continuous Time Filter Time Delay

Figure 10

Impulse Response

$$c_{2i}(t) := \omega_{nF}^{2} \cdot t \cdot 1 \cdot s \cdot \exp(-\omega_{nF} \cdot t)$$



peak at

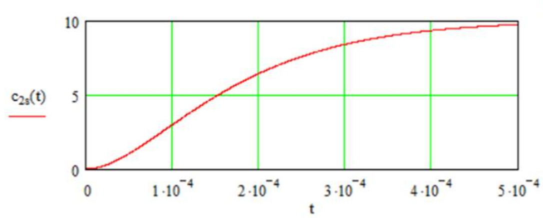
$$c_{2i}(91.1 \cdot \mu \cdot s) = 4037.771$$

Figure 11

Step Response

$$\frac{1}{s} \frac{\Omega_n^{2}}{(s + \Omega_n)^{2}}$$

$$c_{2s}(t) := 10 \cdot \left[1 - \exp(-\Omega_{nF} \cdot t)(1 + \Omega_{nF} \cdot t)\right]$$



$t_y := 354.39 \cdot \mu \cdot s \qquad t_z := 48.452 \cdot \mu \cdot s$

$$1 - (1 + \Omega_{nF} \cdot t_y) \cdot \exp(-\Omega_{nF} \cdot t_y) = 0.9$$

$$1 - (1 + \Omega_{nF} \cdot t_z) \cdot \exp(-\Omega_{nF} \cdot t_z) = 0.1$$

$$t_{r10\_90} := t_y - t_z$$

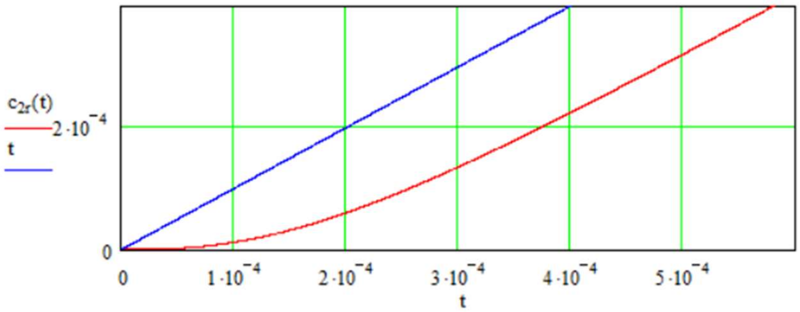
10-90 Rise Time $\quad t_{r10\_90} = 305.938\,\mu \cdot s$

Figure 12

RAMP Response

$$\frac{1}{s^2}\frac{\omega_n^2}{(s+\omega_n)^2} \qquad \textbf{[}\zeta = 1\textbf{]}$$

$$c_{2r}(t) := t - \frac{2}{\omega_{nF}} + \frac{2}{\omega_{nF}}\cdot\exp(-\omega_{nF}\cdot t)\cdot\left(1+\frac{\omega_{nF}\cdot t}{2}\right)$$



$$T_{delay} := \frac{2\cdot\zeta_F}{\omega_{nF}}$$

$$T_{delay} = 182.22\,\mu\cdot s$$

Figure 13

This agrees somewhat with the phase delay calculation for τ_{filt_delay}.

<u>Discrete Time Filter in Frequency Domain</u>

$$\Omega_A(\omega_D) := 2\cdot\tan\left(\frac{\omega_D}{2}\right)$$ Using EQ 14

$$\omega_D(\Omega_a) := \frac{2}{T_s}\cdot\operatorname{atan}\left(\frac{\Omega_A\cdot T_s}{2}\right)$$ Using EQ 15

$$\omega_{\Omega T}(\Omega_a) := \Omega_a\cdot T_s$$ Using EQ 16

--=

Let $\omega_{\Omega T\_start} := 0.470931$ Which is < 2 x ε $\varepsilon = 0.3$ $2\cdot\varepsilon = 0.6$ **Per Cond. 2**

$$\frac{\omega_{\Omega T\_start}}{T_s} = 7064\,\frac{rad}{s}$$ **so Ωa Ts ~ ωd** $$\frac{1}{T_f} = 7064\,\frac{rad}{s}$$

Therefore the Continous Domain startpoint is

$$\Omega_{a\_start} := 7063.95 \cdot \frac{rad}{s} \quad -3 \cdot dB \qquad \text{gain response freq.}$$

Designate the Continous Domain endpoint, as Ω_{a_end}

$$\Omega_{a\_end} := \frac{\omega_{\Omega T\_start}}{T_s} \cdot F_x \quad -6 \cdot dB \qquad \text{gain response freq.}$$

$$\Omega_{a\_end} = 10975.8 \frac{rad}{s}$$

$$\Omega_{a\_end} \cdot T_s = 0.73172$$ Which is > 2 x ε

So prewarping is needed!

Define $k_1 := 1.4, 1.41 .. 1.9$

$$\omega_{d\_end}(k_1) := \omega_{\Omega T\_start} \cdot k_1$$ **Discrete Filter Frequency cutoff**

$$\Omega_{a\_PW} = k_2 \cdot \Omega_{a\_start}$$ **The Prewarp Analog Frequency cutoff to obtain the above Discrete Filter frequency cutoff!**

Discrete cutoff freq $\omega_{dc} = 2 \cdot \operatorname{atan}\left(k_2 \cdot \Omega_{a\_start} \cdot \frac{T_s}{2}\right)$

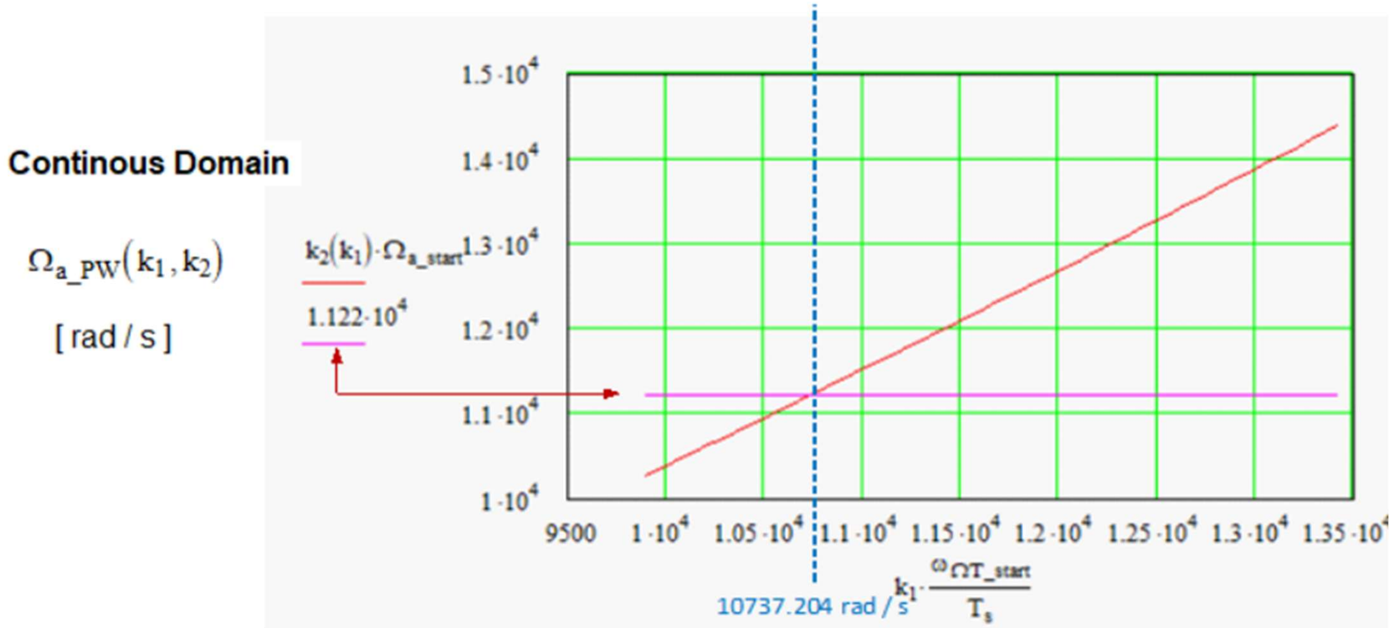
If $\omega_{dc} = \omega_{d\_end}$

Then $$\frac{\omega_{d\_end}(k_1)}{2} = \operatorname{atan}\left(k_2 \cdot \Omega_{a\_start} \cdot \frac{T_s}{2}\right) \qquad \textbf{EQ 24}$$

Solving for k2 $$k_2(k_1) := \frac{2}{T_s \cdot \Omega_{a\_start}} \cdot \tan\left(\frac{\omega_{d\_end}(k_1)}{2}\right) \qquad \textbf{EQ 25}$$

$$\Omega_{a\_PW}(k_1, k_2) := k_2(k_1) \cdot \Omega_{a\_start} \qquad \textbf{EQ 26}$$

$\Omega_{a\_PW}(k_1, k_2)$ is the prewraped frequency



Discrete Domain

Figure 14

Choose k1 and test against the criterion below, until the desired accuracy ([k1 - Fx] / Fx * 100) < 0.2%) is obtained (~ 3-7 tries typically)

Criterion $k_1 = F_y$

$$F_y = \frac{-6\text{dB}\cdot\text{discrete}\cdot\text{frequency}}{-3\cdot\text{dB}\cdot\text{discrete}\cdot\text{frequency}} \qquad \frac{\text{(unknown value)}}{\text{(known value)}}$$

$$F_x = \frac{-6\text{dB}\cdot\text{continuous}\cdot\text{frequency}}{-3\cdot\text{dB}\cdot\text{continuous}\cdot\text{frequency}} \qquad \frac{\text{(known value)}}{\text{(known value)}} \qquad F_x = 1.553774$$

-3 dB frequency of continuous(Ω_a) and discrete (ω_D / Ts) are assumed to be identical = 7063.95 rad / s. This is because Cond. 2 above is met. From Cond. 2 above we can derive –

$$\text{If} \quad \frac{\Omega_a}{2\cdot f_s} \le 0.3 \qquad \text{then} \qquad f_s \ge \frac{\Omega_a}{0.6}$$

$$\text{With} \quad \Omega_a = 7064\cdot\frac{\text{rad}}{\text{s}} \qquad f_s \ge \frac{7064\cdot\frac{\text{rad}}{\text{s}}}{0.6} = 11773.333\cdot\text{Hz}$$

This is why 15 KHz was selected for fs. This makes ε ~ 0.2355.

Trial 1

$k_1 = 1.50$

$k_2 = 1.56565$

Actual

$$\frac{10595.925 \cdot \frac{rad}{s}}{7063.95 \cdot \frac{rad}{s}} = 1.5 \qquad \frac{10595 \cdot \frac{rad}{s}}{6975.35 \cdot \frac{rad}{s}} = 1.519 \qquad F_y = 1.51892$$

$k_1 \neq F_y$

Trial 2

$k_1 = 1.54$

$k_2 = 1.61125$

Actual

$$\frac{10878.483 \cdot \frac{rad}{s}}{7063.95 \cdot \frac{rad}{s}} = 1.54 \qquad \frac{10878 \cdot \frac{rad}{s}}{7170.9 \cdot \frac{rad}{s}} = 1.517 \qquad F_y = 1.51696$$

$k_1 \neq F_y$

Trial 3

$k_1 = 1.52$

$k_2 = 1.58841$

Actual

$$\frac{10737.2 \cdot \frac{rad}{s}}{7064 \cdot \frac{rad}{s}} = 1.52 \qquad \frac{10736 \cdot \frac{rad}{s}}{7073 \cdot \frac{rad}{s}} = 1.51788 \qquad F_y = 1.51788$$

within 0.14 %

$k_1 = F_y$

$$\omega_{d\_end}(1.52) \cdot f_s = 10737.2 \frac{rad}{s}$$ Discrete frequency cutoff

$$\Omega_{a\_PW}(1.52, k_2) = 11220.5 \frac{rad}{s}$$ Continuous prewrap frequency

Analysis Details:

Therefore, design an Analog Filter with a <u>prewarped</u> cutoff of 11220.5 rad / s, to obtain a Digital Filter, which after the Bilateral Transform is applied, has a cutoff of 10737.2 rad/s.

Steps to Design a Digital Filter with a cutoff freq of 11220.5 rad / s, fs = 15 KHz

$$\Omega_c := 11220.5 \cdot \frac{rad}{s}$$

Continuous Domain Filter prototype -- 2nd order $\zeta = 1$

$$\Omega_c \cdot \frac{T_s}{2} = 0.374017$$

$$F_p(S) = \frac{\Omega_c^{\,2}}{S^2 + 2 \cdot \Omega_c \cdot S + \Omega_c^{\,2}}$$

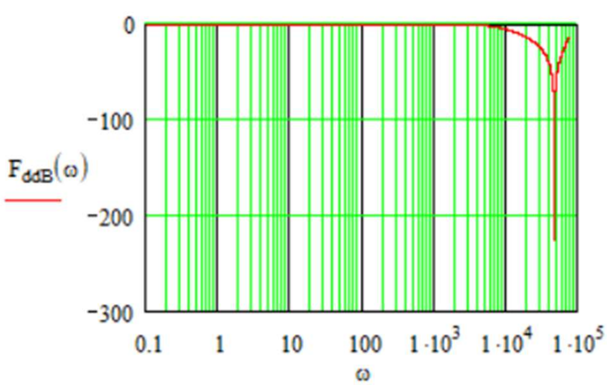
Apply the Bilinear Transform -

Discrete Domain Filter

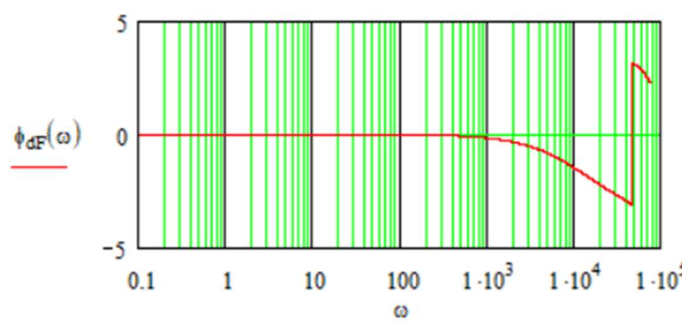
$$F_{dF}(Z) = \frac{\left(\frac{2}{T_s}\right)^2 \cdot 0.374017^2}{\left(\frac{2}{T_s}\right)^2 \cdot \left(\frac{1 - Z^{-1}}{1 + Z^{-1}}\right)^2 + \left[2 \cdot \left(\frac{2}{T_s}\right)^2 \cdot 0.374017 \cdot \left(\frac{1 - Z^{-1}}{1 + Z^{-1}}\right)\right] + \left(\frac{2}{T_s}\right)^2 \cdot 0.374017^2} \qquad \textbf{EQ 27}$$

$$F_d(\omega) := \frac{Z(\omega)^2 + 2 \cdot Z(\omega) + 1}{13.496 \cdot Z(\omega)^2 - 12.297 \cdot Z(\omega) + 2.801} \qquad \textbf{EQ 28}$$

$$F_{ddB}(\omega) := 20 \cdot \log\left(\left|F_d(\omega)\right|\right) \qquad \phi_{dF}(\omega) := \arg\left(F_d(\omega)\right)$$



Discrete Time Filter Amplitude Response
Figure 15

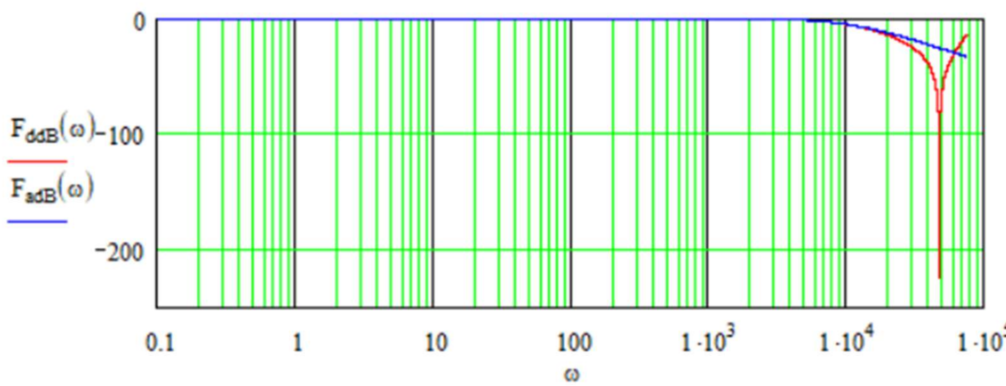


Discrete Time Filter Phase Response
Figure 16

$$F_{ddB}\left(7073 \cdot \frac{rad}{s}\right) = -3\,dB \qquad \phi_{dF}\left(7073 \cdot \frac{rad}{s}\right) = -65.43\,deg$$

Very Good Match

$$F_{ddB}\left(10736.0 \cdot \frac{rad}{s}\right) = -6.02\,dB \qquad \phi_{dF}\left(10736 \cdot \frac{rad}{s}\right) = -89.99\,deg$$

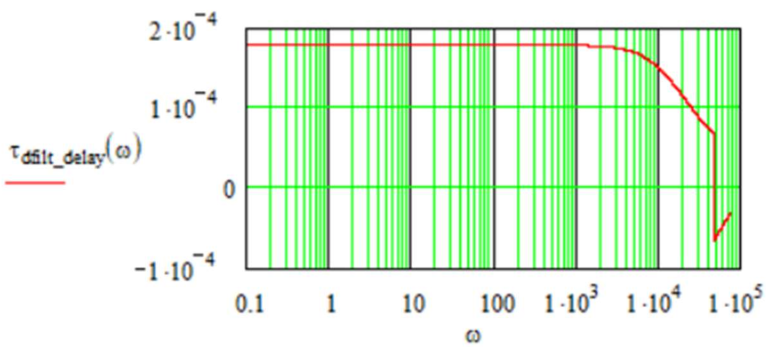


Continuous and Discrete Filter Comparison
Figure 17

$k_1 \sim F_y$

So we are finished!

$$\tau_{dfilt\_delay}(\omega) := \frac{-\phi_{dF}(\omega)}{\omega}$$



$$\tau_{dfilt\_delay}\left(1 \cdot \frac{rad}{s}\right) = 178\,\mu \cdot s$$

Equivalent to analog filter!

Discrete Domain Time Response

Test Signals

$k := 0, 1 .. 50$

$STEP(k) := 10 \cdot \Phi(k)$ Step Amplitude

$RAMP(k) := k \cdot T_s \cdot \Phi(k) \cdot s^{-1}$ Ramp Slope

$y_{0\_init} := 0$ **"Test Signals"**

$y_0(k) := f_s \cdot 1 \cdot s \cdot \delta(k, 2)$ Note the "Amplitude" is 1/Ts.

$y_1(k) := \left(y_{0\_init} + STEP(k-2)\right) \cdot \Phi(k-2)$

$y_2(k) := \left(y_{0\_init} + RAMP(k-2)\right) \cdot \Phi(k-2)$

$$F_d(\omega) := \frac{1 + 2 \cdot Z(\omega)^{-1} + Z(\omega)^{-2}}{13.496 - 12.297 \cdot Z(\omega)^{-1} + 2.801 \cdot Z(\omega)^{-2}}$$ **EQ 29**

$$U(Z) \cdot \left(13.496 - 12.297 \cdot Z^{-1} + 2.801 \cdot Z^{-2}\right) = Y(Z) \cdot \left(1 + 2 \cdot Z^{-1} + Z^{-2}\right)$$ **EQ 30**

$z_k := y_0(k)$ $u_k := z_k$ $y_k := z_k$ **Initalize Vectors**

$$U(Z)\left(1 - 0.911159 \cdot Z^{-1} + 0.207543 \cdot Z^{-1}\right) = Y(Z) \cdot \left(0.074096 + 0.148192 \cdot Z^{-1} + 0.074096 \cdot Z^{-2}\right)$$ **EQ 31**

Impulse Response Difference Equation (Includes a shift of k by +2 for MCAD)

$$u_{k+2} := 0.074096 \cdot y_0(k+2) + 0.148192 \cdot y_0(k+1) + 0.074096 \cdot y_0(k) + 0.911159 \cdot u_{k+1} - 0.207543 \cdot u_k$$

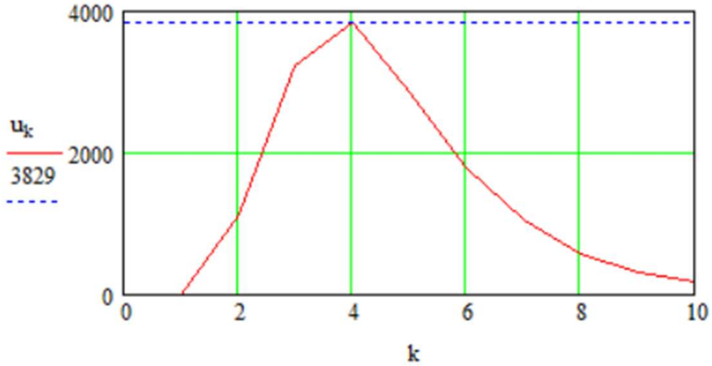


Figure 18

Note there is a delay of 2 samples

first peak at $u_3 = 3235.579$ $3 \cdot T_s = 200\,\mu \cdot s$

$$3 \cdot T_s - 2 \cdot T_s = 66.667\,\mu \cdot s$$

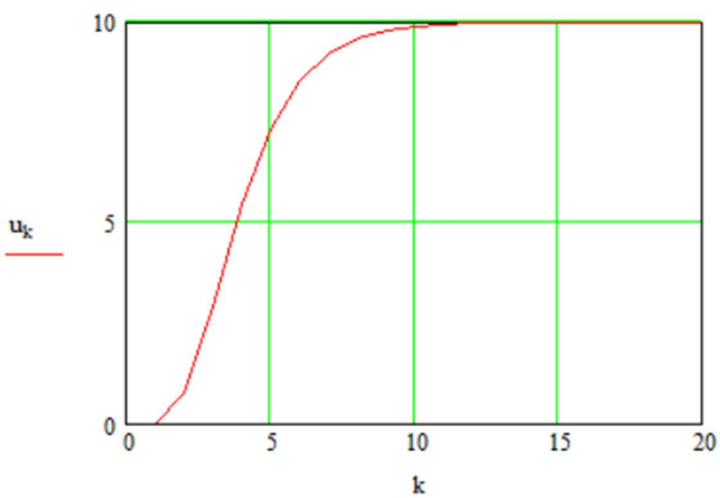
true peak at

$$u_4 = 3828.895 \quad 4 \cdot T_s = 266.667\,\mu \cdot s$$

$$4 \cdot T_s - 2 \cdot T_s = 133.333\,\mu \cdot s$$

Step Response

$$u_{k+2} := 0.074096 \cdot y_1(k+2) + 0.148192 \cdot y_1(k+1) + 0.074096 \cdot y_1(k) + 0.911159 \cdot u_{k+1} - 0.207543 \cdot u_k$$



Fiure 19

$u_2 = 0.741$ $u_7 = 9.197$

$$(6.9 - 2.3) \cdot T_s = 306.667\,\mu \cdot s$$

RAMP Response

$$u_{k+2} := 0.074096 \cdot y_2(k+2) + 0.148192 \cdot y_2(k+1) + 0.074096 \cdot y_2(k) + 0.911159 \cdot u_{k+1} - 0.207543 \cdot u_k$$

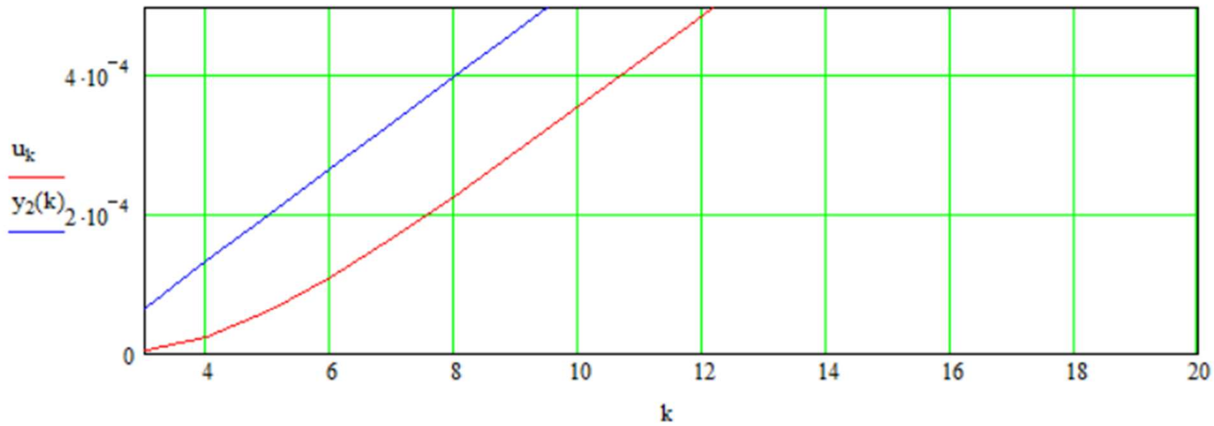


Figure 20

$$(10 - 7.27) \cdot T_s = 182 \mu \cdot s \qquad t_{Dfilt\_delay} := 182 \cdot \mu \cdot s$$

Part 6 – Inclusion of Time Delays; Filter and Zero Order Hold (ZOH).

What about θ? Time Delay is the sum of the Zero Order Hold Delay + the Filter Delay In either the continuous or discrete models.

Zero Order Hold

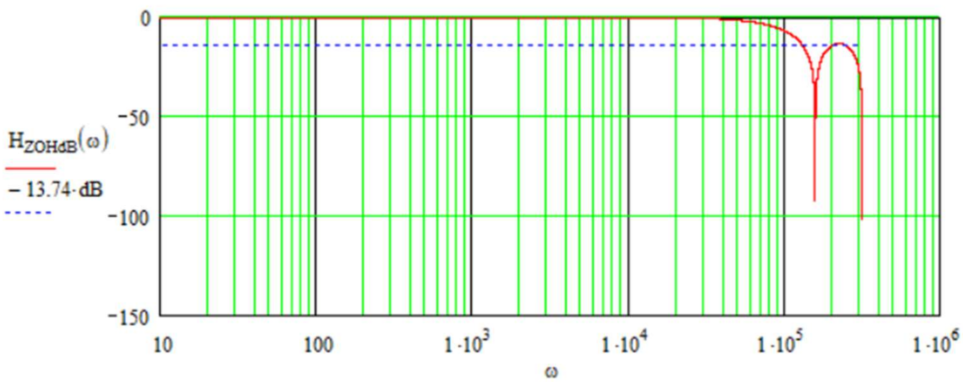
$$H_{ZOH}(\omega) := \frac{1 - \exp(-j \cdot \omega \cdot T_s)}{j \cdot \omega \cdot T_s} \qquad \textbf{EQ 32}$$

$$\text{sinc}(0) = 1 \qquad \text{sinc}(x) = \frac{\sin(\pi \cdot x)}{\pi \cdot x}$$

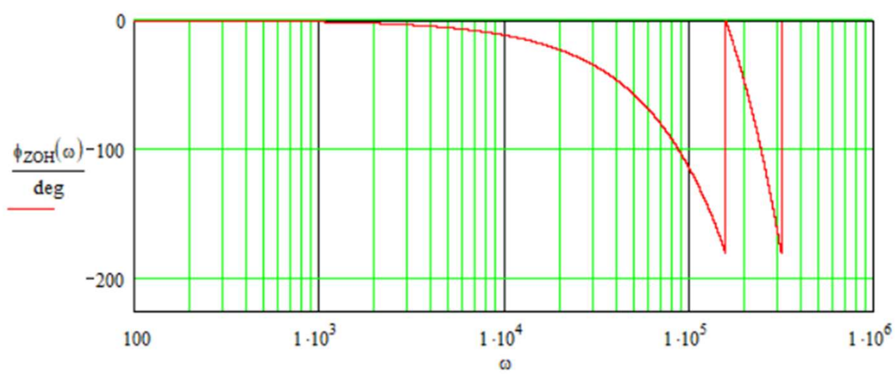
Alternate Form

$$H_{ZOH}(\omega) = \exp\left(-j \cdot \frac{T_s}{2} \cdot \omega\right) \cdot \text{sinc}\left(\frac{\omega \cdot T_s}{2}\right) \qquad \textbf{EQ 33}$$

$$H_{ZOHdB}(\omega) := 20 \cdot \log\left(\left|H_{ZOH}(\omega)\right|\right) \qquad \phi_{ZOH}(\omega) := \arg\left(H_{ZOH}(\omega)\right)$$



Zero Order Hold Amplitude Response
Figure 21



Zero Order Hold Phase Response
Figure 22

$$\tau_{ZOH\_delay} = \frac{-\phi_{ZOH}(\omega)}{\omega} = \frac{T_s}{2}$$

$$\theta := \frac{T_s}{2} + T_{delay} \qquad \frac{T_s}{2} = 33.333\,\mu \cdot s \qquad T_{delay} = 182.22 \times 10^{-6}\,s$$

$$\theta = 215.55\,\mu \cdot s$$

Note that the continuous domain value is used. As has been shown, there is no significant difference between the two domains frequency and time response. Therefore, continuous domain analysis is used going forward. Note that the discrete difference equations are what is used by an MCU / DSP.

Design Details

Nyquist Bandwidth $\frac{\pi}{T_s} = 47123.89\frac{\text{rad}}{\text{s}}$ Desired Closed Loop Bandwidth $\frac{1}{\tau_c} = 22\frac{\text{rad}}{\text{s}}$

$$K_P = 0.008599 \qquad k := 4.376575$$

$$T_I = 1.719\,\text{milli}\cdot\text{s} \qquad T_D = 1415.636\,\mu\cdot\text{s} \qquad T_f = 141.564\,\mu\cdot\text{s}$$

$$G_{CL}(S) = \frac{G_C(S)\cdot F(S)\cdot G_P(S)}{1 + G_C(S)\cdot F(S)\cdot G_P(S)} = \frac{K_P\cdot\left(1 + \frac{1}{S\cdot T_I} + S\cdot T_D\right)\cdot F(S)\cdot G_P(S)}{1 + K_P\cdot\left(1 + \frac{1}{S\cdot T_I} + S\cdot T_D\right)\cdot F(S)\cdot G_P(S)} \qquad \textbf{EQ 34}$$

As noted above, we will ignore the F(S) and zero hold contributions to amplitude but include their <u>phase contribution</u> which is time delay:

Therefore, the **Dominant Pole Response** is given by:

Expanding

Velocity Closed Loop $G_{CLspd}(S) = \frac{G_C(S)\cdot G_P(S)}{1 + G_C(S)\cdot G_P(S)}$ **EQ 35**

$$G_{CLspd}(S) = \frac{\left[\frac{K_P}{S\cdot T_I}\cdot\left(S^2\cdot T_D\cdot T_I + S\cdot T_I + 1\right)\right]\cdot\frac{k\cdot\exp(-\theta\cdot S)}{S^2\cdot\tau^2 + S\cdot 2\cdot\zeta\cdot\tau + 1}}{1 + \left[\frac{K_P}{S\cdot T_I}\cdot\left(S^2\cdot T_D\cdot T_I + S\cdot T_I + 1\right)\right]\cdot\frac{k\cdot\exp(-\theta\cdot S)}{S^2\cdot\tau^2 + S\cdot 2\cdot\zeta\,\tau + 1}}$$

$$= \frac{\frac{K_P\cdot k}{T_I\cdot\tau^2}\cdot\left(S^2\cdot T_I\cdot T_D + S\cdot T_I + 1\right)\cdot\exp(-\theta\cdot S)}{S^3 + S^2\cdot\frac{2\cdot\zeta}{\tau} + S\cdot\frac{1}{\tau^2} + \frac{K_P\cdot k}{T_I\cdot\tau^2}\cdot\left[-S^3\cdot T_D\cdot T_I\cdot\theta + S^2\cdot\left(T_D\cdot T_I - T_I\cdot\theta\right) + S\cdot\left(T_I - \theta\right) + 1\right]}$$

$$= \frac{\frac{K_P \cdot k}{T_I \cdot \tau^2} \cdot \left(S^2 \cdot T_I \cdot T_D + S \cdot T_I + 1\right) \cdot \exp(-\theta \cdot S)}{S^3 \cdot \left(1 - T_D \cdot \theta \cdot \frac{K_P \cdot k}{\tau^2}\right) + S^2 \cdot \left[\frac{2 \cdot \zeta}{\tau} + \left(T_D - \theta\right) \cdot \frac{K_P \cdot k}{\tau^2}\right] + S \cdot \left[\frac{1}{\tau^2} + \left(1 - \frac{\theta}{T_I}\right) \cdot \frac{K_P \cdot k}{\tau^2}\right] + \frac{K_P \cdot k}{T_I \cdot \tau^2}} \qquad \textbf{EQ 36}$$

$$1 - T_D \cdot \theta \cdot \frac{K_P \cdot k}{\tau^2} = 0.99528 \qquad \frac{2 \cdot \zeta}{\tau} + \left(T_D - \theta\right) \cdot \frac{K_P \cdot k}{\tau^2} = 724.95805 \frac{1}{s}$$

$$\left[\frac{1}{\tau^2} + \left(1 - \frac{\theta}{T_I}\right) \cdot \frac{K_P \cdot k}{\tau^2}\right] = 4.24514 \times 10^5 \frac{1}{s^2} \qquad \frac{K_P \cdot k}{T_I \cdot \tau^2} = 8.99889 \times 10^6 \frac{1}{s^3}$$

$$D_1(S) := S^3 \cdot 0.99528 + S^2 \cdot 724.95805 + S \cdot 4.24514 \cdot 10^5 + 8.99889 \cdot 10^6$$

$$G_{PIDC}(\omega) := K_P \cdot \left(1 + \frac{1}{j \cdot \omega \cdot T_I} + \frac{j \cdot \omega \cdot T_D}{1 + j \cdot \omega \cdot \gamma}\right)$$

$$D_1(S) \text{ solve}, S \rightarrow \begin{pmatrix} -353.19821236703065457 - 535.01191924650586355 \cdot i \\ -353.19821236703065457 + 535.01191924650586355 \cdot i \\ -21.999654761156117107 \end{pmatrix}$$

$$N_1(S) = S^2 \cdot \left(\frac{K_P \cdot k \cdot T_D}{\tau^2}\right) + S \cdot \left(\frac{K_P \cdot k}{\tau^2}\right) + \frac{K_P \cdot k}{T_I \cdot \tau^2}$$

$$\frac{K_P \cdot k}{\tau^2} \cdot T_D = 21.896 \frac{1}{s} \qquad \frac{K_P \cdot k}{\tau^2} = 1.54671 \times 10^4 \frac{1}{s^2} \qquad \frac{K_P \cdot k}{T_I \cdot \tau^2} = 8.99889 \times 10^6 \frac{1}{s^3}$$

$$N_1(S) := S^2 \cdot 21.896 + S \cdot 1.54671 \cdot 10^4 + 8.99889 \cdot 10^6$$

$$N_1(S) \text{ solve}, S \rightarrow \begin{pmatrix} -353.19464742418706613 - 535.01105189435039631 \cdot i \\ -353.19464742418706613 + 535.01105189435039631 \cdot i \end{pmatrix}$$

LT

$$g(t) \quad \longleftrightarrow \quad G(S)$$

LT

$$g(t-\theta) \quad \longleftrightarrow \quad G(S)\cdot\exp(-\theta S)$$

Closed Loop
Step Response

$$G(S) = \frac{1}{S}\cdot\frac{s_1\cdot s_2\cdot s_3}{(S+s_1)\cdot(S+s_2)\cdot(S+s_3)}\cdot\frac{(S+s_4)\cdot(S+s_5)}{s_4\cdot s_5}$$

Poles and Zeroes are as follows:

$s_1 := (353.198 + 535.012j)\cdot\frac{rad}{s}$ $\quad s_2 := \overline{s_1}$ $\quad s_3 := 21.999655\cdot\frac{rad}{s}$

$s_4 := (353.1951 + 535.011j)\cdot\frac{rad}{s}$ $\quad s_5 := \overline{s_4}$

Partial Fraction Expansion

$$\frac{N(S)}{D(S)} = \frac{A_0}{S} + \frac{A_1}{(S+s_1)} + \frac{A_2}{(S+s_2)} + \frac{A_3}{(S+s_3)}$$

$$\mathbf{A_0 = \frac{S\cdot N(S)}{D(S)}} \quad \Big| \quad \mathbf{S = 0}$$

$$\frac{s_1\cdot s_2\cdot s_3}{s_4\cdot s_5}\cdot\frac{(0+s_4)\cdot(0+s_5)}{(0+s_1)\cdot(0+s_2)\cdot(0+s_3)} = 1$$

$A_0 := 1$

$$A_1 = \frac{(s + s_1)\cdot N(S)}{D(S)} \Bigg| \quad S = -s_1$$

$$A_1 := \frac{s_1 \cdot s_2 \cdot s_3}{s_4 \cdot s_5} \cdot \frac{(-s_1 + s_4)\cdot(-s_1 + s_5)}{(-s_1)\cdot(-s_1 + s_2)\cdot(-s_1 + s_3)}$$

$$A_1 = 0 + 0i$$

$$A_2 = \frac{(S + s_2)\cdot N(S)}{D(S)} \Bigg| \quad S = -s_2$$

$$A_2 := \frac{s_1 \cdot s_2 \cdot s_3}{s_4 \cdot s_5} \cdot \frac{(-s_2 + s_4)\cdot(-s_2 + s_5)}{(-s_2)\cdot(-s_2 + s_1)\cdot(-s_2 + s_3)}$$

$$A_2 = 0 - 0i$$

$$A_3 = \frac{(S + s_3)\cdot N(S)}{D(S)} \Bigg| \quad S = -s_3$$

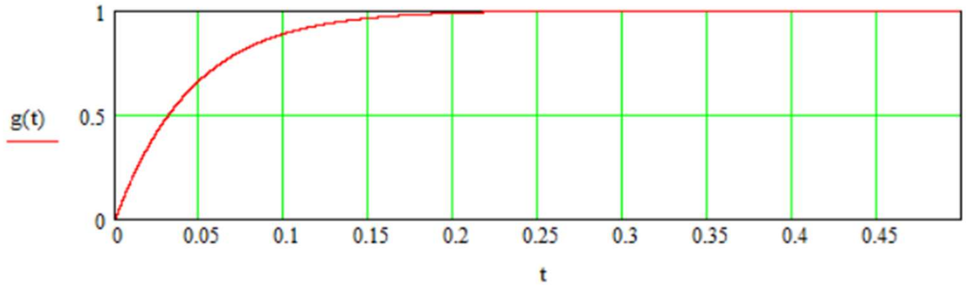
$$A_3 := \frac{s_1 \cdot s_2 \cdot s_3}{s_4 \cdot s_5} \cdot \frac{(-s_3 + s_4)\cdot(-s_3 + s_5)}{-s_3\cdot(-s_3 + s_1)\cdot(-s_3 + s_2)} \qquad A_3 = -1$$

$$\lambda := \mathrm{Re}(A_1) \qquad \lambda = 0$$

$$\delta := \mathrm{Im}(A_1) \qquad \delta = 0$$

$$t := 0\cdot s, 0.00001\cdot s \,..\, 1\cdot s$$

$$g(t) := \left[\left[1 - \exp(-\sigma_1\cdot t)\cdot\left(-2\cdot\lambda\cdot\cos(\omega_1\cdot t) + -2\cdot\delta\cdot\sin(\omega_1\cdot t)\right)\right] + A_3\cdot\exp(-\sigma_3\cdot t)\right]\cdot\Phi(t) \qquad \textbf{EQ 37}$$



Velocity Closed Loop Step Response

Figure 23

Effective Bandwidth = 22 rad / s

Frequency Domain

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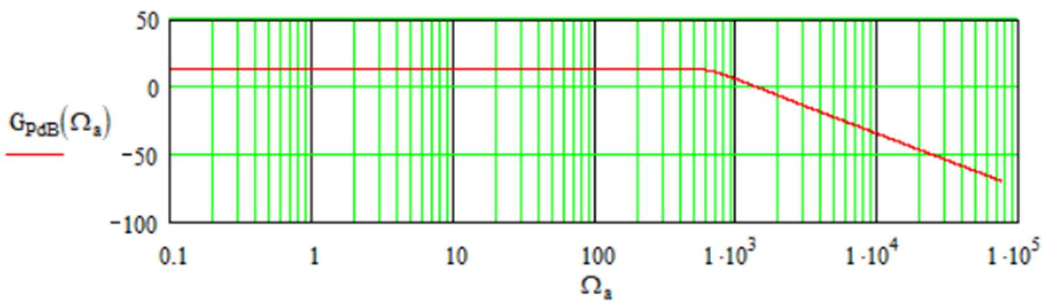
Process Portion

$$G_P(\Omega_a) := \frac{k}{\tau^2 \cdot (j \cdot \Omega_a)^2 + 2 \cdot \zeta \cdot \tau \cdot (j \cdot \Omega_a) + 1}$$

"From EQ 9 - less θ"

$$G_{PdB}(\Omega_a) := 20 \cdot \log\left(\left|G_P(\Omega_a)\right|\right) \qquad \phi_P(\Omega_a) := \arg\left(G_P(\Omega_a)\right)$$



$$\Omega_r := \Omega_n \cdot \sqrt{1 - 2 \cdot \zeta^2}$$

Amplitude Plot of GPdB(ω)

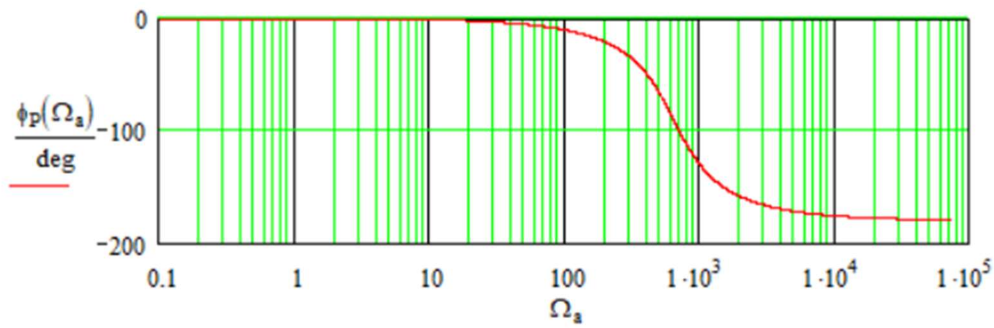
Figure 24

3 dB bandwidth

$$G_{PdB}\left(776.011 \cdot \frac{rad}{s}\right) - G_{PdB}\left(0.1 \cdot \frac{rad}{s}\right) = -3\,dB$$

Peak

$$G_{PdB}(\Omega_r) = 13.551\,dB$$



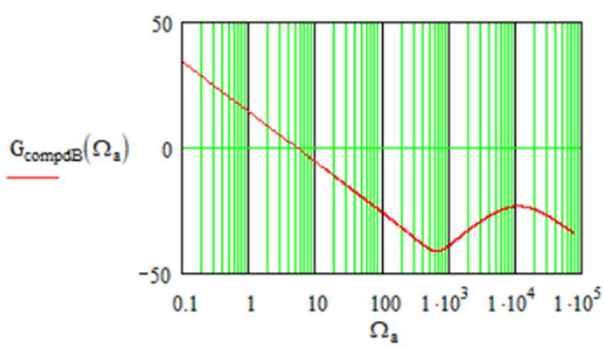
Phase Plot of $G_P(\omega)$
Figure 25

$$\phi_P(\Omega_n) = -90\,\text{deg}$$

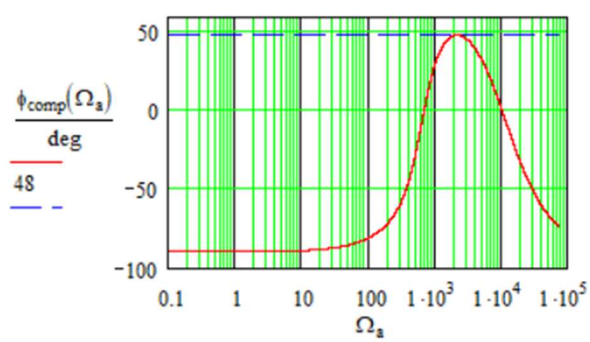
Compensation plus Filter:

Adding the effect of the filter on the compensation yields –

$$G_{compdB}(\Omega_a) := G_{CdB}(\Omega_a) + F_{adB}(\Omega_a) \qquad \phi_{comp}(\Omega_a) := \phi_C(\Omega_a) + \phi_{aF}(\Omega_a)$$



Amplitude Plot of $G_{CdB}(\omega)$ +$F_{dB}(\omega)$
Figure 26



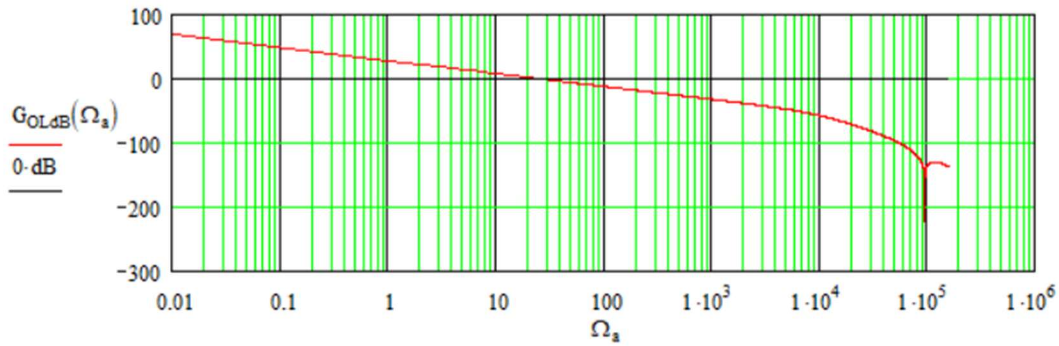
Phase Plot of $G_C(\omega)$ +$F(\omega)$
Figure 27

"Exact" Open Loop Gain is obtained from EQ 39 below:

$$G_{OL}(\Omega_a) := G_C(\Omega_a)\cdot F_a(\Omega_a)\cdot H_{ZOH}(\Omega_a)\cdot G_P(\Omega_a) \qquad \textbf{EQ 39}$$

$$G_{OLdB}(\omega) := 20\cdot\log(|G_{OL}(\omega)|) \qquad \text{Amplitude}$$

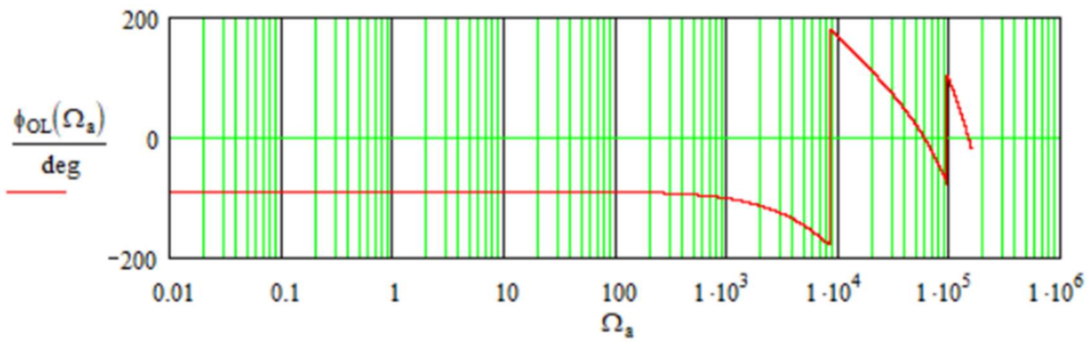
$$\phi_{OL}(\omega) := \arg(G_{OL}(\omega)) \qquad \text{Phase}$$



Amplitude Plot of $G_{OLdB}(\omega)$

Figure 28

<u>Gain Margin – 55.6 dB</u> @ -180 deg Phase Shift is at 829 rad/s (~ 131.9 Hz)



Phase Plot of $G_{OL}(\omega)$

Figure 29

$$\omega_c = \text{crossover\_frequency}(0 \cdot dB)$$

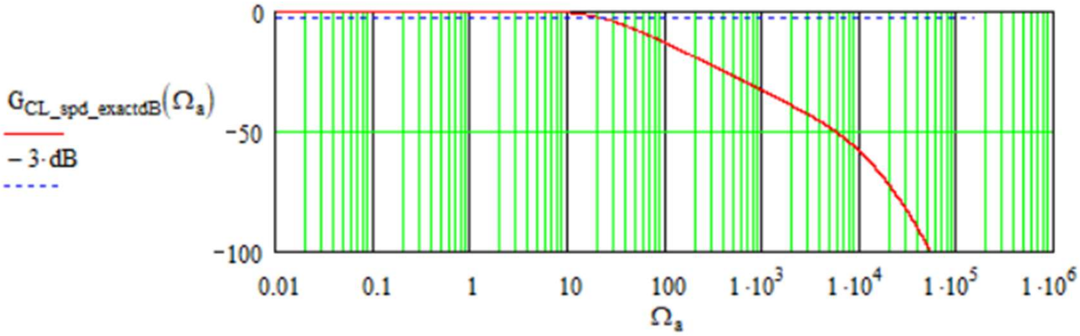
$$\omega_c = 22 \cdot \frac{rad}{s} \quad \text{approximately}$$

<u>Phase Margin ~ 89.7 deg</u>

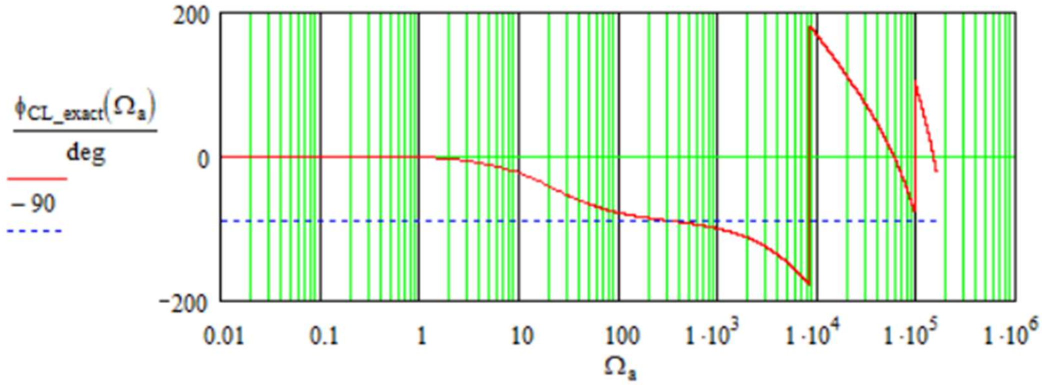
$$G_{CL\_spd\_exact}(\Omega_a) := \frac{G_{OL}(\Omega_a)}{1 + G_{OL}(\Omega_a)} \qquad \textbf{EQ 40}$$

$$G_{CL\_spd\_exactdB}(\Omega_a) := 20 \cdot \log\left(\left|G_{CL\_spd\_exact}(\Omega_a)\right|\right) \qquad \text{Amplitude}$$

$$\phi_{CL\_exact}(\Omega_a) := \arg\left(G_{CL\_spd\_exact}(\Omega_a)\right) \qquad \text{Phase}$$



Closed Loop Velocity Amplitude Plot
Figure 30



Closed Loop Velocity Phase Plot
Figure 31

Closed Loop Summary

$$G_{CL\_spd\_exactdB}\left(21.948 \cdot \frac{rad}{s}\right) = -3\,dB \qquad \Omega_{3dBf} := 21.948 \cdot \frac{rad}{s}$$

$$\phi_{CL\_exact}(\Omega_{3dBf}) = -45.204\,deg$$

Target 3dB frequency is 22 rad / s, overshoot was to be 0 % (per **EQ 10** – 1 / τ_c).

- There is only a 0.24 % error in 3dB frequency.
- On target for overshoot (0 %) – Refer to Figure 23.
- No setpoint filter is required.
- However, a 15 KHz sampling frequency was used.

Part 9 – Changing The Closed Loop Bandwidth

What about changing the servo loop's bandwidth?
Note that the proportional term of EQ 12 is the only one which is a function of the closed loop bandwidth and is proportional to 1 / τ_c.

$$K_P = \frac{2\cdot\zeta\cdot\tau}{k\cdot\left(\tau_c + \theta\right)} \qquad \textbf{EQ 12a}$$

Therefore, as the integral and derivative times are a function of only the process (motor) alone - (from EQ 12b and EQ 12c)

$$T_I = 2\cdot\zeta\cdot\tau \quad \textbf{EQ 12b} \qquad T_D = \frac{\tau}{2\cdot\zeta} \quad \textbf{EQ 12c}$$

And the filter is a function of

$T_f = \alpha\cdot T_D$; α is selected to be a constant 0.1

The open loop gain is adjusted up or down depending on whether the bandwidth is to be increased or decreased.

An example will help clarify this effect.

Example 1.
Change the desired closed loop bandwidth to (~ 7 Hz). That is

$$\Omega_{c\_new} := 44\cdot\frac{rad}{s}$$

$$\tau_{c\_new} := \frac{1}{\Omega_{c\_new}}$$

$$\tau_{c\_new} = 22.727\ milli\cdot s$$

$$K_{P\_new} := \frac{2\cdot\zeta\cdot\tau}{k\cdot\left(\tau_{c\_new} + \theta\right)}$$

$$K_{P\_new} = 0.017117$$

Let

$$\Delta G_{OL} := \frac{K_{P\_new}}{K_P} \qquad \textbf{EQ 41}$$

$$\Delta G_{OLdB} := 20 \cdot \log\left(\left|\Delta G_{OL}\right|\right) \qquad \textbf{EQ 42}$$

ΔG_{OL} is > 1 when $K_{P\_new} > K_p$ **Increasing bandwidth**

and < 1 when $K_{P\_new} < K_P$ **Decreasing bandwidth**

and = 1 when $K_{P\_new} = K_P$ **No change to bandwidth**

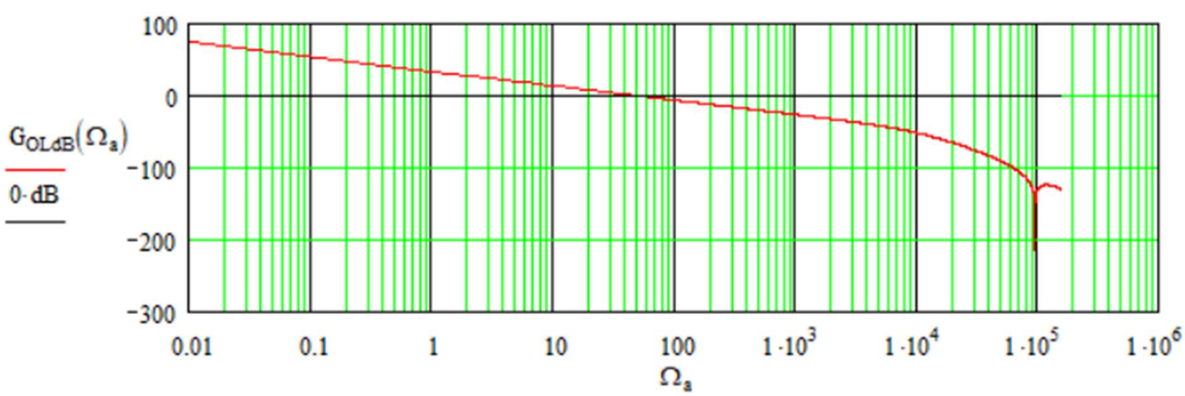
For this example, $K_{P\_new} > K_P$

$\Delta G_{OL} = 1.991$

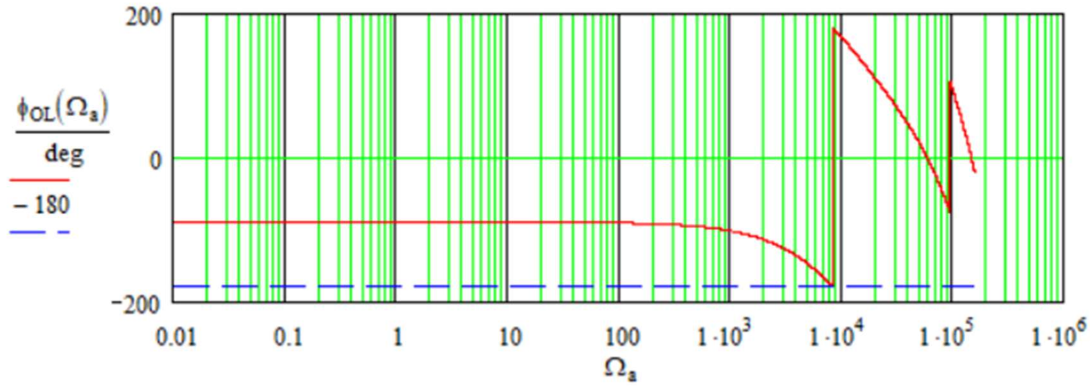
$\Delta G_{OLdB} := 5.98 \cdot dB$

Therefore, we should expect the Gain Margin to be reduced from 55.6 dB to 49.6 dB (55.6- 6 dB)
$G_{OLdB}(\Omega_{co\_new}) = 0dB$ when $\Omega_{co\_new} = 1.991 \times 22$ rad/s = 43.8 rad / s.

The crossover frequency is found to be 43.585 rad / s.



Open Loop Gain with Ωc_new = 44 rad / s
Figure 32

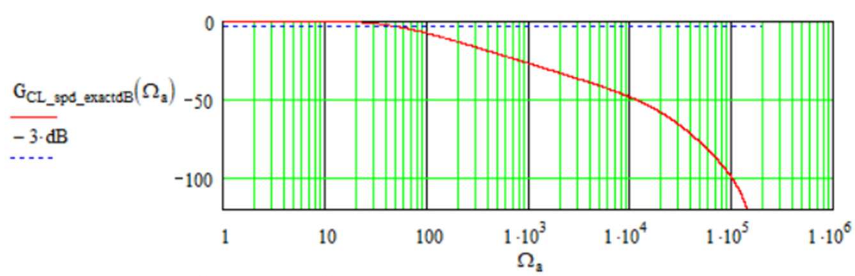


Open Loop Phase with Ωc_new = 44 rad / s
Figure 33

$$G_{OLdB}\left(43.5849\cdot\frac{rad}{s}\right) = -0\,dB \qquad \phi_{OL}\left(43.6\cdot\frac{rad}{s}\right) = -90.538\,deg$$

$$G_{OLdB}\left(0.829\cdot 10^{4}\cdot\frac{rad}{s}\right) = -49.6\,dB \qquad \phi_{OL}\left(0.829\cdot 10^{4}\cdot\frac{rad}{s}\right) = -179.96009\,deg$$

$$GM = 49.6dB \qquad PM = 89.46\cdot deg$$



$$G_{CL\_spd\_exactdB}\left(43.896\cdot\frac{rad}{s}\right) = -3\,dB \qquad \Omega_{3dBf} := 43.896\cdot\frac{rad}{s}$$

Closed Loop Gain with Ωc_new = 44 rad /s
Figure 34

The actual closed loop 3 dB bandwidth is 43.896 rad / s (7.145 Hz) . Which is in error by less than 0.25 % (-0.236%). Rise Time (10-90 %) = 2.1972 / 43.896 rad / s ~ 50 msec.

- **Rise time of cascaded blocks [from Wikipedia reference [14] Rise time - Wikipedia]**

$$t_{ro10\_90} = \sqrt{t_{rs}^{\ 2} + t_{r1}^{\ 2} + t_{r2}^{\ 2}}$$

With

$$t_{rs} := 0 \cdot s \qquad t_{r1} := \frac{2.1972}{s_3} \qquad t_{r2} := 2.1972 \cdot \theta$$

$$t_{r1} := 0.04994 \cdot s \qquad t_{r2} := 0.00474 \cdot s$$

$$t_{r10\_90} := \sqrt{t_{r1}^{\ 2} + t_{r2}^{\ 2}} \qquad t_{r10\_90} = 50.16 \times 10^{-3} s$$

Pretty close match. Within 0.32 %.

What if the motor had real poles? Then

EQ 9 is replaced by EQ 36

$$G_P(S) = \frac{k \cdot \exp(-\theta \cdot S)}{(S \cdot \tau_a + 1) \cdot (S \cdot \tau_b + 1)} \qquad \textbf{EQ 43} \qquad \zeta > 1$$

After following the Direct Synthesis approach of Part 3 one gets –

$$K_P = \frac{(\tau_a + \tau_b) \cdot \tau}{k \cdot (\tau_c + \theta)} \quad \textbf{EQ 44a} \qquad T_I = \tau_a + \tau_b \quad \textbf{EQ 44b} \qquad T_D = \frac{\tau_a \cdot \tau_b}{(\tau_{a.} + \tau_b)} \quad \textbf{EQ 44c}$$

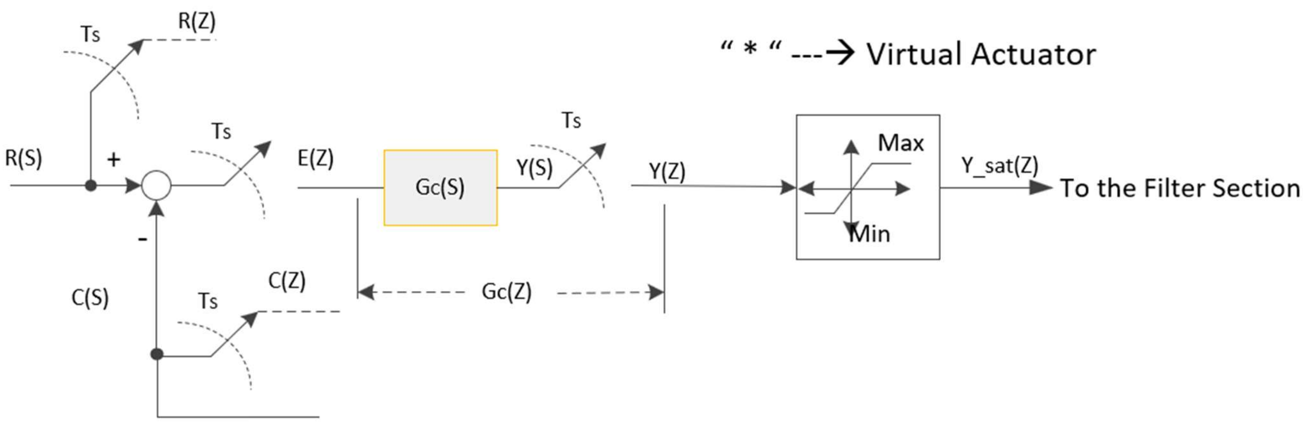
Where τ_a and τ_b are the motors real poles.
Refer to reference **[7]** equations (25), (26), (27) and (28).

Otherwise, the design steps are identical.

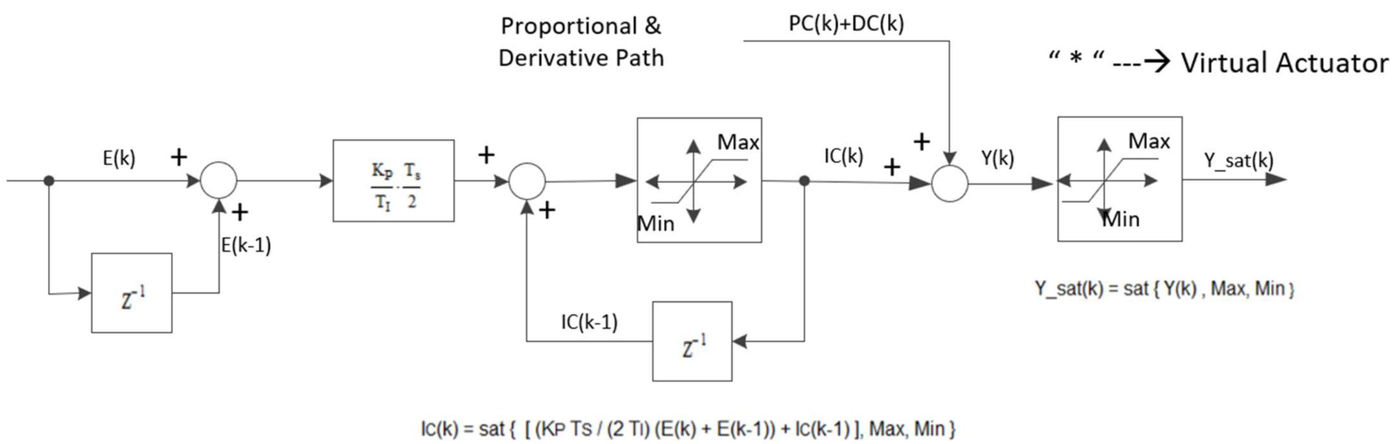
Part 10 – Integral Anti-windup

It is important to provide a method to prevent Integral Windup. It was not discussed in the document **[8] York, Frederick, PI Controller Design using a Limited Span Method**, but is important in that instance as well.

The Block Diagram of Figure 35 depicts the section of the PID with a limiter ("Virtual Actuator") and Figure 36 present one method of integral anti-windup implementation.



Discrete PID Section
Figure 35



Integral Path, One Method of Anti-windup Implementation
Figure 36

Definition of sat function:
Where LL (Lower Limit) = **Min** level; and UL (Upper Limit) = **Max** level.

$$y = \mathrm{sat}(x, LL, UL) \quad \begin{cases} y = x & \text{when } LL < x < UL \\ y = LL & \text{when } x \le LL \\ y = UL & \text{when } x \ge UL \end{cases}$$

First let us define the upper and lower limits. Assume that a direction signal dir is used to differentiate clockwise (CW) from counterclockwise (CCW) motor rotation. As a sinusoidal drive is intended to be used, a two's complement value is to be generated using the dir signal. If the magnitude is given by a 9-bit ADC value. then10 bits must be used as a minimum. A16/32-bit value (signed short or signed long integer) would be expected in actual ues.

Per the motor specifications – CW from lead side, (dir = 0 phase sequence is A-B-C) is positive and the CCW (dir = 1, phase sequence is A-C-B) is negative (dir = 1).

Now we determine the maximum Vin value (or VMOT) for our limit as follows:

Design operation for the motor example –

$\tau_{nom} := 0.18 \cdot N \cdot m$ $\qquad \tau_{max} := 0.216 \cdot N \cdot m$ Working Torque - By Design

$\omega_{out\_nom} := 26.18 \cdot \frac{rad}{s}$ Working upper motor angular velocity

<u>Subscript definition (abbreviated by sub in the equations)</u>: nom = nominal operating value and max = max operating value and limit is the absolute operating boundary. Also hot = after steady state temperature rise is achieved and p = peak and rms = root mean square.

Motor parameters derived from datasheet (sine / sine):

$$I_{\phi\phi p\_nom} := \frac{2}{\sqrt{3}} \cdot \frac{\tau_{nom}}{K_{Ep}} \qquad I_{\phi\phi p\_nom} = 0.91967\ A$$

$$I_{\phi\phi rms\_nom} := \frac{\tau_{nom}}{\sqrt{3} \cdot K_{Erms}} \qquad I_{\phi\phi rms\_nom} = 0.65033\ A_{rms}$$

$$I_{\phi\phi p\_max} := \frac{2}{\sqrt{3}} \cdot \frac{\tau_{max}}{K_{Ep}} \qquad I_{\phi\phi p\_max} = 1.10361\ A$$

$$I_{\phi\phi rms\_max} := \frac{\tau_{max}}{\sqrt{3} \cdot K_{Erms}} \qquad I_{\phi\phi rms\_max} = 0.7804\ A$$

$$degC := K \qquad \theta_T := 5.2 \cdot \frac{degC}{W} \qquad R_{\phi\phi\_nom} := R$$

Resistance at the steady state temperature due to the operating current.

$$R_{\phi\phi\_sub\_hot} = R_{\phi\phi} + \Delta R_{\phi\phi\_sub\_hot} \qquad \textbf{EQ 45}$$

Resistance change relationship:

Copper resistance percentage rise due to temperature = Hot Resistor power loss times the thermal impedance.

$$\frac{\Delta R_{\phi\phi\_sub\_hot}}{R_{\phi\phi}} \times \frac{100\%}{0.393\%} \cdot degC \quad = \quad I_{\phi\phi rms\_sub}{}^{2} \cdot R_{\phi\phi\_nom} \cdot \theta_T \cdot \left(1 + \frac{\Delta R_{\phi\phi\_sub\_hot}}{R_{\phi\phi}}\right)$$

Let

$$l_0 := \frac{100\%}{0.393\%} \cdot degC \qquad\qquad l_1 := R_{\phi\phi\_nom} \cdot \theta_T$$

$$\frac{\Delta R_{\phi\phi\_sub\_hot}}{R_{\phi\phi\_nom}} \cdot l_0 = I_{\phi\phi rms\_sub}{}^{2} \cdot l_1 + I_{\phi\phi rms\_sub}{}^{2} \cdot l_1 \cdot \frac{\Delta R_{\phi\phi\_sub\_hot}}{R_{\phi\phi\_nom}}$$

$$R_{\phi\phi\_sub\_hot} := R_{\phi\phi\_nom} \cdot \left[1 + \frac{\left(I_{\phi\phi rms\_sub}{}^{2} \cdot l_1\right)}{\left(l_0 - I_{\phi\phi rms\_sub}{}^{2} \cdot l_1\right)}\right] \qquad \textbf{EQ 46}$$

Using EQ 46 results in -

$$R_{\phi\phi\_sub\_hot} = 5.885\,\Omega$$

$$R_{\phi\phi\_max\_hot} = 6.02\,\Omega$$

The motor voltage input voltage can be found by using EQ 47.

$$V_{MOTrms\_nom} := \omega_{out\_nom} \cdot \left(K_{Erms} + \frac{D}{K_{Trms}} \cdot R_{\phi\phi\_sub\_hot}\right) + \frac{\tau_{nom}}{K_{Trms}} \cdot R_{\phi\phi\_sub\_hot} \qquad \textbf{EQ 47}$$

From EQ 47 one finds

$$V_{MOTrms\_nom} = 8.059\,V_{rms} \qquad\qquad I_{\phi\phi rms\_nom} = 0.65\,A_{rms}$$

$V_{MOTrms\_nom} \cdot I_{\phi\phi rms\_nom} = 5.241\,W$ **Nominal**

$V_{MOTrms\_max} = 8.93\,V_{rms}$ $\quad$ $I_{\phi\phi rms\_max} = 0.78\,A_{rms}$

$V_{MOTrms\_max} \cdot I_{\phi\phi rms\_nom} = 5.808\,W$ **Max**

For a locked rotor condition ($\omega_{out} = 0$ rad/s)
Let

$$V_{MOTrms\_limit} = V_{MOTrms\_max} = \frac{\tau_{limit}}{K_{Trms}} \cdot R_{\phi\phi\_limit\_hot}$$

Try

$$I_{\phi\phi rms\_limit} := 1.291 \cdot A_{rms}$$

Then

$$R_{\phi\phi\_limit\_hot} = 6.92\,\Omega$$

$$\tau_{limit} := I_{\phi\phi rms\_limit} \cdot \sqrt{3}\left(K_{Erms}\right)$$

$$\tau_{limit} = 0.357\,N \cdot m$$

$V_{MOTrms\_limit} = 8.933\,V_{rms}$ $\quad$ $I_{\phi\phi rms\_limit} = 1.291\,A$

$V_{MOTrms\_limit} \cdot I_{\phi\phi rms\_limit} = 11.53\,W$ **Limit**

Motor operating temperature at the limit can be found as follows:

$$T_{H\_max} = 11.532 \cdot W \cdot \theta_T = 60 \cdot degC$$

$T_{A\_max} := 55 \cdot degC$ $\quad$ Assume an upper ambient temperature

$$T_{H\_max} := 60 \cdot degC$$

$$T_{Mot} := T_{A\_max} + T_{H\_max}$$

$T_{Mot} = 115\,degC$ $\quad$ is OK

As the maximum allowable winding temperature is 155 degC, there is no issue with this motor voltage with a locked rotor.

$$8.933 \cdot V_{rms} \cdot \sqrt{2} = 12.633\,V \qquad \text{Peak Voltage}$$

Assume that we can control each phase-to-phase amplitude to 9 bits of resolution. If the maximum amplitude is Vpmax = 17.25 Vp, then

$$V_{pmax} := 17.25 \cdot V \qquad ADC := 9\text{bit}$$

$$\frac{12.633 \cdot V}{V_{pmax}} \cdot 2^{ADC} = 375 \qquad \text{rounded-up}$$

Showing only the significant 10 bits --

UL for CW rotation	0101110111b	375
LL for CCW rotation	1010001001b	−375

Motor specs give

$$17.4V_p \le V_{MOT}$$

If we have equal positive and negative limits, then UL = - LL. So, UL = 375 and LL = -375. If the limits are identical regardless of dir, then we can use the magnitude and ignore the sign. If there are different positive and negative limits, then using the separate UL and LL values is the only option.

If a commercially available integrated circuit such as Texas Instruments DRV10983 or equivalent is used, then the component will include additional safety limits as well. These include undervoltage lockout (UVLO), stall detection (phase current limiting) and over-temperature shutdown among others. Note that an adjustment of θ is required (additional 160 μs to be added) since the analog speed sample period, t_{SAM}, is 320 μs.

This still leaves one the responsibility of handling the integral windup.

Part 11 – Summary

<u>Summary of Steps taken for this design</u>:

1. Generate Specifications and Select a Motor.
2. Use the Motor Transfer Function of EQ 7.
3. Solve EQ 7 for the Natural Frequency and Damping Factor, Ω_n and ζ.
4. Determine the PID Parameters K_P, T_I and T_D using EQ 12a, EQ 12b and 12c if $\zeta <= 1$ or EQ 37a, EQ 37b and EQ 37c if ζ is > 1.
5. Find the value $T_f = \alpha * T_D$ by selecting a value for α (usually set to 0.1). This determines where the 3dB frequency of the Filter is to be.
6. Find Y = Tf / Fx (F_X = 1.553774) which establishes the Ωn for this filter (1 / Y), the 6 dB frequency as $\zeta_F = 1$.
7. Select a sample rate which is

   $$f_s \geq \frac{\Omega_{a\_start}}{0.6} = \frac{1}{0.6T_f}$$
8. As an MCU or DSP will most likely be used for the PID and Filter, first, establish the Discrete PID design factors. Refer to EQ 21.
9. Next, establish the Discrete Filter parameters using EQ 25 and EQ 26.
   This is an iterative approach which leads to the necessary Continuous Domain Prototype prewrapped cutoff frequency (- 6 dB) which will yield the appropriate Discrete Filter cutoff (-6 dB) after the Bilateral Transform is applied.
10. Find the exact Open Loop (EQ 39) and Closed Loop Gain (EQ 340). Check design against the original specifications.
11. Add Integrator Anti Windup to the design.

Part 12- List of References

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