PI Velocity Controller Design using a Limited Span Method

This document is intended to expand upon the work done by *Dave Wilson* of Texas Instruments in the Teaching Your PI Controller to Behave series of articles. This was an excellent set of 10 presentations which deals with the selection of gain constants used in a Cascaded PI (Proportional Integral) Servo Controller. Dave Wilson's work reduces the PI coefficients to a single parameter which I will refer to as Span (Dave referred to this as the damping factor, δ) Dave's work is a generalization of the Symmetrical Optimization (SO) Tuning derived by Kessler (1958) which was modified by Voda and Landau (1995). The technique is used to maximize the phase margin for a given closed loop system.

My effort is to show how one can improve a control system's setpoint and disturbance response, when employing SO with a Limited Span Method (LSM).

The Method proposed herein, uses a relationship between the prefiltered velocity loop bandwidth ($\omega_{3dB}f$) and span, χ but in a limited span range ($2 <= \chi < 2.95$).

That Block Diagram shown below is a current controller embedded within a servo velocity feedback loop. See Figure 1 below.

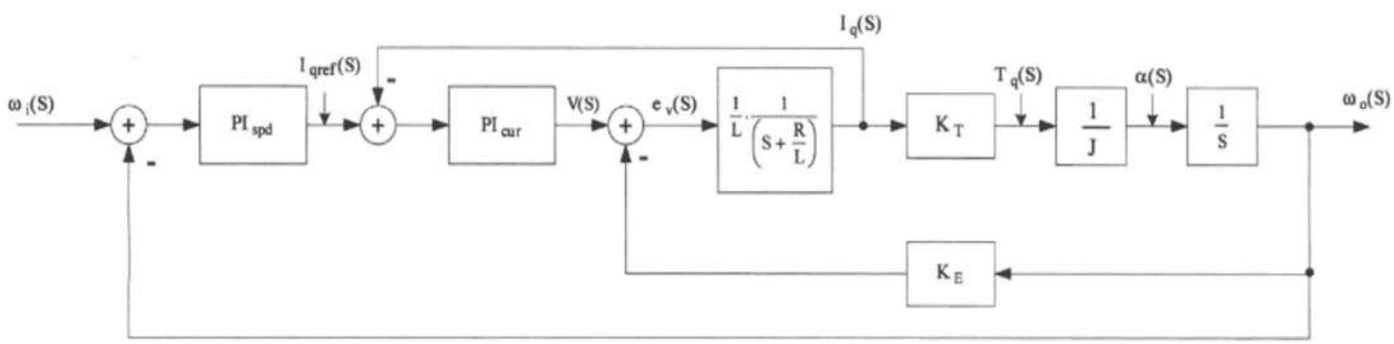


Figure 1

Simplified Cascaded Velocity / Current Loop Servo System

Symbols:

Symbol	Interpretation	Units
V	Voltage (Potential difference)	Volts (V)
I	Current	Amps (A)
R	Resistance	ohms (Ω)
L	Inductance	Henries (Wb / A(turn)
τ	Torque	Newton Meter (Nm)
τ_m, τ_e	Time constants	Seconds (s)
t	Time	Seconds (s)
ω	Angular velocity	(rad/s)
α	Angular acceleration	(rad/s/s)
ɵ	Angle mechanical	(radians or degrees)
P	Number of magnetic poles	(P/2 = Number of pole pairs)
J	Moment of Inertia (Total)	(kg m^2)
K_{px}	Proportional Gain per subscript	(series PI Controller)
K_{IX}	Integral Gain term per subscript	(series PI Controller)
K_T	Total torque constant	various dimensions
χ -	Symmetry Factor or Span	(Same as δ in Dave Wilson's Articles)
ω_{3dB}	Velocity Closed Loop Bandwidth	rad/s (or Hertz)
ω_{3dBf}	Velocity Closed Loop Bandwidth but with an input prefilter to cancel ω_z (zero) and/or ω_1 (pole)	rad/s (or Hertz)

Why try to better a good technique? [The enemy of good is better]. Because when chi (χ), is greater than three (3), the closed loop suffers from less open loop gain and a dominant real pole (ω_1) is introduced which impacts disturbance response time. However, as we will show, there is also a slight peaking in the sensitivity function when the open loop crossover (0 dB) frequency is exceeded.

There are five (5) parts to this document:

Part 1 - Summary of Equation Derivations from Dave Wilson's Teaching Your PI Controller to Behave. "Standard Symmetrical / Limited Span Symmetry Method".

Part 2 - Analysis of velocity closed loop parameters when Prefiltering is used.

Part 3 – Range Adjustment for Limited Span Method.

Part 4 - Torque Disturbance Impact and Comparison of Design Methods.

Part 5– Conclusions.

Part 1 - Summary of Equation Derivations from Dave Wilson's Teaching Your PI Controller to Behave:

Definitions –

Note the following changes to the nomenclature from Dave Wilson's articles.
Herein:
Kpcur (from Ka) current controller proportional gain.
KIcur (from Kb) current controller integral gain.
Kpspd (from Kc) speed controller proportional gain.
Kispd (from Kd) speed controller integral gain.
X, chi (from δ, delta) for the symmetry factor (referred to as by Dave Wilson as the damping factor)

Though damping and symmetry are strongly related they are not the same, as will be shown later.

Series PI Controller Transfer Function (subscript j denotes the controller type, e.g., cur for current, and spd for speed).

This form is assumed throughout.

$$PI(S) = \frac{K_{pj} \cdot K_{Ij}}{S} + K_{pj} \qquad \text{EQ 1}$$

Motor Voltage to Motor Current Transfer Function

$$\frac{I(S)}{V(S)} = \frac{\frac{1}{R}}{\left(1 + \frac{L}{R} \cdot S\right)} \qquad \text{EQ 2}$$

Let

$$K = \frac{K_T}{J}$$

$$K_A = K \cdot K_{pspd} \cdot K_{Ispd}$$

ω_p is the current loop Bandwidth

ω_D is the highest open loop pole - not ωp.

ω_0 is the open loop crossover frequency.

ω_z is the open / closed loop zero frequency

The following was obtained from Dave Wilson's Analysis.

Current Controller

$$j := \sqrt{-1}$$

$$G_{ol\_cur}(S) = \left(\frac{K_{pcur} \cdot K_{Icur}}{S} + K_{pcur}\right) \cdot \frac{1}{L} \cdot \frac{1}{\left(S + \frac{R}{L}\right)}$$

$$= \frac{K_{pcur}}{L} \cdot \left(\frac{S + K_{Icur}}{S}\right) \cdot \frac{1}{\left(S + \frac{R}{L}\right)} \qquad \textbf{EQ 3.}$$

$$G_{cl\_cur}(S) = \frac{\frac{K_{pcur}}{L} \cdot \left(\frac{S + K_{Icur}}{S}\right) \cdot \frac{1}{\left(S + \frac{R}{L}\right)}}{1 + \frac{K_{pcur}}{L} \cdot \left(\frac{S + K_{Icur}}{S}\right) \cdot \frac{1}{\left(S + \frac{R}{L}\right)}}$$

$$= \frac{K_{pcur}}{L} \cdot \left[\frac{\left(S + K_{Icur}\right)}{S^2 + S \cdot \left(\frac{R}{L} + \frac{K_{pcur}}{L}\right) + \frac{K_{pcur} \cdot K_{Icur}}{L}}\right]$$

With $K_{Icur} = \frac{R}{L}$

$$G_{cl\_cur}(S) = \frac{K_{pcur}}{L} \cdot \frac{\left(S + \frac{R}{L}\right)}{\left(S + \frac{R}{L}\right) \cdot \left(S + \frac{K_{pcur}}{L}\right)} \qquad \textbf{EQ 4}$$

Therefore

$$G_{cl\_cur}(S) = \frac{K_{pcur}}{L} \cdot \frac{1}{\left(S + \frac{K_{pcur}}{L}\right)} \qquad \textbf{EQ 5.}$$

Speed Controller With Transportation Lag (Delay Time) τ_D

$$G_{ol\_spd}(S) = \left(\frac{K_{pspd} \cdot K_{Ispd}}{S} + K_{pspd}\right) \cdot \frac{K_{Tp}}{S \cdot J} \cdot \frac{\exp(-S \cdot \tau_D)}{\left[1 + \frac{S}{\left(\frac{K_{pcur}}{L}\right)}\right]}$$

$$= \frac{K_{pspd} \cdot K_{Ispd} \cdot \left(1 + \frac{S}{K_{Ispd}}\right)}{S} \cdot \frac{K_{Tp}}{S \cdot J} \cdot \frac{\exp(-S \cdot \tau_D)}{\left[1 + \frac{S}{\left(\frac{K_{pcur}}{L}\right)}\right]}$$

If the Transportation Lag (delay) frequency (1/TD) is much less than the Current Loop bandwith ω_p

With $\omega_p = \frac{K_{pcur}}{L}$

Define

$\omega_D = \frac{1}{\tau_D}$ $\omega_D < \omega_p$ and using the approximation

For ω << $\frac{1}{\tau_D}$ $\exp(-S \cdot \tau_D)$ ~ $1 - S \cdot \tau_D$ ~ $\frac{1}{1 + S \cdot \tau_D}$

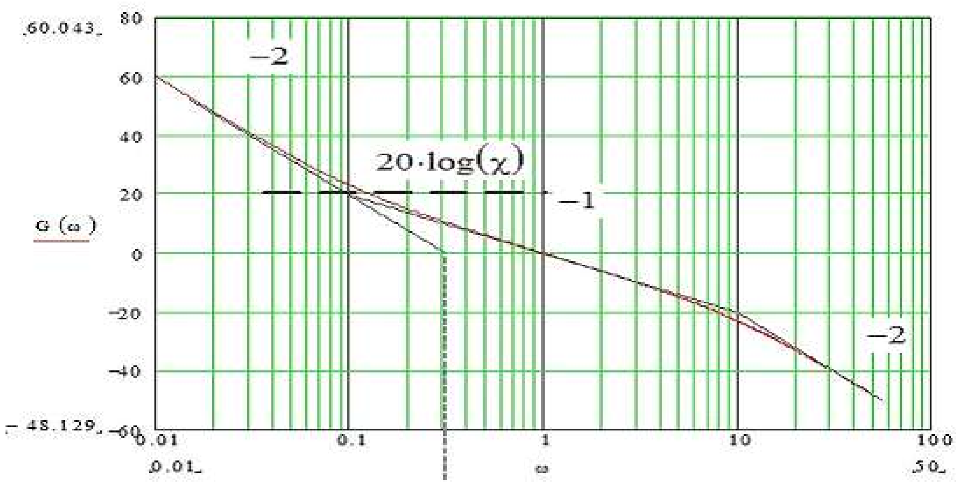
$$G_{ol\_spd}(S) = \frac{K_{Tp} \cdot K_{pspd} \cdot K_{Ispd}}{J} \cdot \frac{1}{S^2} \cdot \frac{\left(1 + \frac{S}{K_{Ispd}}\right)}{\left(1 + \frac{S}{\omega_D}\right)} \qquad \textbf{EQ 6}$$

Let $$K_A = \frac{K_{Tp} \cdot K_{pspd} \cdot K_{Ispd}}{J} \qquad \textbf{EQ 7}$$

$$G_{ol\_spd}(S) = \frac{1}{\left(\frac{S^2}{K_A}\right)} \cdot \frac{\left(1 + \frac{S}{K_{Ispd}}\right)}{\left[1 + \frac{S}{(\omega_D)}\right]} \qquad \textbf{EQ 8}$$

Let $S = j \cdot \omega$ Form is that of a "Lead" Compensator

$$G_{ol\_spd}(j \cdot \omega) = \frac{1}{\left[\frac{(-\omega)^2}{K_A}\right]} \cdot \frac{\left(1 + \frac{j \cdot \omega}{K_{Ispd}}\right)}{\left[1 + \frac{j \cdot \omega}{(\omega_D)}\right]} \qquad \textbf{EQ 9}$$



$$\sqrt{\frac{K_A}{\chi}} \qquad \sqrt{K_A} \qquad \qquad \chi^{\frac{3}{2}} \cdot \sqrt{K_A}$$

$$\frac{\omega_c}{\chi} \qquad \omega_c \qquad \chi \cdot \omega_c$$

geometric mean for peak phase compensation

$$\omega_z = K_{Ispd} \qquad \omega_z \qquad \omega_c = \sqrt{\chi \cdot K_A} \qquad \omega_D \qquad \omega_D = \frac{1}{\tau_D}$$

Open Loop Response Approximation
Figure 2

$$G_{ol\_spd}(j\cdot\omega) = \frac{1}{\left(\frac{-\omega^2}{K_A}\right)}\cdot\frac{\left(1 + j\cdot\frac{\omega}{\omega_z}\right)}{\left(1 + j\cdot\frac{\omega}{\omega_D}\right)}$$

At the Crossover, ω_o (0 dB)

$$\left|G_{ol\_spd}(j\cdot\omega_o)\right| = \frac{K_A}{\omega_o^2}\cdot\frac{\sqrt{1 + \left(\frac{\omega_o}{K_{Ispd}}\right)^2}}{\sqrt{1 + \left[\frac{\omega_o}{(\omega_D)}\right]^2}}$$

That is

using $$K_{Ispd} = \frac{\omega_D}{\chi^2} \qquad\qquad \frac{\omega_D}{\chi^2} = \omega_z$$

and $$\omega_o = \chi\cdot K_{Ispd}$$

$$\left|G_{ol\_spd}(j\cdot\omega_o)\right| = \frac{K_A}{(\chi\cdot K_{Ispd})^2}\cdot\frac{\sqrt{1 + \chi^2}}{\sqrt{1 + \chi^{-2}}}$$

$$\left|G_{ol\_spd}(j\cdot\omega_o)\right| = \frac{K_A}{(\chi\cdot K_{Ispd})^2}\cdot\frac{\chi\cdot\sqrt{1 + \chi^2}}{\sqrt{1 + \chi^2}}$$

when $\left|G_{ol\_spd}(j\cdot\omega)\right| = 1$ then $\frac{K_A}{\chi\cdot K_{Ispd}^2} = 1$

let $$K = \frac{K_{Tp}}{J}$$

$$\frac{K}{\chi} \cdot \frac{K_{pspd}}{K_{Ispd}} = 1$$

$$K_{pspd} = \frac{\chi \cdot K_{Ispd}}{K} = \frac{\omega_D}{\chi \cdot K}$$

Summary

$$K_{Icur} := \frac{R}{L}$$ (rad/s)

*

$$K_{pcur} = L \cdot \omega_p$$ (ohms)

$$K_{Ispd} = \sqrt{\frac{K_A}{\chi}} = \omega_z$$ (rad/s)

$$K_{pspd} = \frac{\chi}{K} \cdot K_{Ispd} = \frac{\omega_D}{\chi \cdot K}$$ (A/rad/s)

*

$$5 \cdot \omega_D < \frac{\omega_p}{L} < \frac{\omega_{sample\_rate}}{10}$$

Table 1

Derived PI Gain Constant Values

Other Important Constants / Units

$$K = \frac{K_T}{J} \quad \left(\frac{\frac{N \cdot m}{A_p}}{kg \cdot m^2} \right) \quad \left(\frac{\frac{rad}{s^2}}{A_p} \right)$$ Acceleration / Amp peak

$$K_A = K \cdot K_{pspd} \cdot K_{Ispd} \quad \left(\frac{rad^2}{s^2} \right)$$ Overall Gain

Closed Loop Speed Loop is found from

$$G_{cl\_spd}(S) = \frac{\left[\frac{K_A\cdot\left(\frac{\omega_D}{\omega_z}\right)}{S^2}\cdot\frac{(S+\omega_z)}{(S+\omega_D)}\right]}{1+\frac{K_A\cdot\left(\frac{\omega_p}{\omega_z}\right)}{S^2}\cdot\frac{(S+\omega_z)}{(S+\omega_D)}}$$

GH / (1+GH) H = 1

$$G_{cl\_spd} = \frac{K_A\cdot\left(\frac{\omega_D}{\omega_z}\right)\cdot(S+\omega_z)}{S^2\cdot(S+\omega_D)+K_A\cdot\left(\frac{\omega_D}{\omega_z}\right)\cdot(S+\omega_z)}$$

EQ 10

$$G_{cl\_spd}(S) = K_A\cdot\left(\frac{\omega_D}{\omega_z}\right)\cdot\frac{(S+\omega_z)}{S^3+S^2\cdot\omega_D+S\cdot K_A\cdot\left(\frac{\omega_D}{\omega_z}\right)+K_A\cdot\omega_D}$$

EQ 11

Or

$$G_{cl\_spd}(S) = \frac{\omega_D^{\,2}}{\chi}\cdot\frac{(S+\omega_z)}{S^3+S^2\cdot\omega_D+S\cdot\frac{\omega_D^{\,2}}{\chi}+\frac{\omega_D^{\,3}}{\chi^3}}$$

EQ 12

$$S = j\omega$$

$$G_{cl\_spd}(j\cdot\omega) = \frac{\left(1+j\frac{\omega}{\omega_z}\right)}{\left(1+j\cdot\frac{\omega}{\omega_1}\right)\cdot\left(1+j\cdot\frac{\omega}{\omega_2}\right)\cdot\left(1+j\cdot\frac{\omega}{\omega_c}\right)}$$

EQ 13

ω_1, ω_2 and ω_c are the roots of the Characteristic Equation (denominator).
Note that for χ >= 3.0 all the roots are real.

Closed Loop Summary:

Note that for any $\chi > 1$, the damping factor, ζ is a piecewise linear function of the span, χ.

$\zeta(\chi) = 0.5\,\chi - 0.5$. EQ 14

For X > 3 , $\zeta > 1$

There are three real poles and one zero.

Poles

$$\omega_1(\chi) = \frac{\omega_D}{2\cdot\chi}\cdot\left(-1 + \chi - \sqrt{\chi^2 - 2\cdot\chi - 3}\right) \qquad \text{EQ 15}$$

$$\omega_2(\chi) = \frac{\omega_D}{2\cdot\chi}\cdot\left(-1 + \chi + \sqrt{\chi^2 - 2\cdot\chi - 3}\right) \qquad \text{EQ 16}$$

$$\omega_c(\chi) = \frac{\omega_D}{\chi} \qquad \text{EQ 17}$$

The estimated 3 dB bandwidth for $\chi >= 4$ with no prefilter applied. (To within 2.5 %)

$$\omega_{3dBe}(\chi) = \frac{\omega_D}{\chi^2}\cdot(2 + \chi) \qquad \text{EQ 18}$$

ζ is the damping ratio.

ω_{3dB} is the closed loop 3 dB bandwidth frequency.

The location of the zero is given by: $\omega_z = \omega_D / \chi^2$ EQ 19

For $\chi = 3$, $\zeta = 1$

There are three real and equal poles $\omega_1 = \omega_c = \omega_2 = \omega_D / 3$.
One real zero at $\omega_z = \omega_D / 3^2$.

Three dB bandwidth for $\chi = 3$ is $\omega_{3dB} = 0.547\ \omega_D$. When using an optimum Lag-Lead prefilter (cancelling both the zero and one pole) the filtered 3 dB bandwidth is given by $\omega_{3dBf} = 0.6423\ (\omega_D / 3) = 0.2141\ \omega_D$.

For $\chi < 3$ 'Limited Span', $\zeta < 1$
There is one real pole and a complex conjugate pole pair and one real zero.
(Note that $\sigma = \alpha = \zeta\ \omega_n$)

$$\sigma(\chi) = \frac{\omega_D}{2\chi}(\chi - 1) \qquad \text{EQ 20}$$

$$\omega_1(\chi) := \sigma(\chi)\cdot\left[1 - j\cdot\frac{\sqrt{-\chi^2 + 2\cdot\chi + 3}}{(\chi - 1)}\right] \qquad \textbf{EQ 21}$$

$$\omega_2(\chi) := \sigma(\chi)\cdot\left[\frac{1 + j\cdot\sqrt{-\chi^2 + 2\cdot\chi + 3}}{(\chi - 1)}\right] \qquad \textbf{EQ 22}$$

$$\omega_c(\chi) := \frac{\omega_D}{\chi} \qquad \textbf{"EQ 17"} \qquad \text{No difference}$$

Note that EQ 15 gives the same result as EQ 21, and EQ 16 the same as EQ 22, and vice versa!
Just a different form.

The 3 dB frequency (no prefilter) can be closely approximated (within 2.2 %) by EQ 23 below:

$$\omega_{3dBe}(\chi) \sim 1.68\cdot\omega_n(\chi) = 1.68\cdot\omega_c(\chi) = 1.68\cdot\frac{\omega_D}{\chi} \qquad \text{EQ 23}$$

The damping frequency can be found from $\sigma(\chi)\cdot\left|\mathrm{Im}(\omega_1(\chi))\right|$ which is

$$\omega_d(\chi) = \frac{\omega_D}{2\cdot\chi}\cdot\sqrt{-\chi^2 + 2\cdot\chi + 3}$$

The location of the zero is given by: $\omega_z = \omega_D / \chi^2$ "EQ 19"

Phase Compensation for all χ.

$$\phi_M(\omega) = \operatorname{atan}\left(\frac{\omega}{\omega_z}\right) - \operatorname{atan}\left(\frac{\omega}{\omega_D}\right) \qquad \text{EQ 24}$$

Peak occurs when $\omega = \omega_c = \omega_0$ and is found from

$$\omega_c = \sqrt{\omega_1(\chi)\cdot\omega_2(\chi)} = \sqrt{\omega_z\cdot\omega_D} = \omega_0$$

$$\phi_{Mpeak}(\omega_c) = \operatorname{asin}\left(\frac{\frac{\omega_D}{\omega_z} - 1}{\frac{\omega_D}{\omega_z} + 1}\right) = \operatorname{atan}\left(\frac{\frac{\omega_D}{\omega_z} - 1}{2\sqrt{\frac{\omega_D}{\omega_z}}}\right) \qquad \text{EQ 25}$$

Or more simply--

$$\phi_{Mpeak} = \operatorname{atan}(\chi) - \operatorname{atan}\left(\frac{1}{\chi}\right) = \operatorname{atan}(2\cdot\zeta + 1) - \operatorname{atan}\left(\frac{1}{2\cdot\zeta + 1}\right) \qquad \text{EQ 26}$$

Notation

Argument χ will be used for both Standard and Limited Span Symmetrical usage.

Examples – $\omega_2(\chi)$, $\omega_1(\chi)$, $\omega_c(\chi)$ and $K_A(\chi)$, and $\zeta_.(\chi)$

For any χ, we find -

$$K_A(\chi) = \frac{\omega_D^{\,2}}{\chi^3} \qquad \text{EQ 27}$$

Define $u(\chi) = \omega_{3dBf}(\chi) / \omega_D$ for the bandwidth ratio**.**

It was discovered that the relative 3 dB bandwidth ($u(\chi)$) of a prefiltered (Lag filter whose pole cancels the zero at ω_z) velocity loop can be found using the following expression: *Computation should always yield an error of less 1.0 %.*

Limited Span

By limited span is meant restricting χ to values between 2 and 2.95.

Note that only a simple low pass prefilter is required to cancel the zero of the transfer function.

For the range (0.175 ω_D) < ω_{3dBf}(χ) <= (0.5 ω_D). Later (in Part 3) we will see later how to adjust (shift) this range.

With a limited span, 2 <= χ <= 2.95.

The relative 3 dB bandwidth as a function of span, u(χ) is given by:

$$u(\chi) := -0.0917\cdot\chi^3 + 0.9103\cdot\chi^2 - 3.1412\cdot\chi + 3.8743 \qquad \text{EQ 28} \qquad < 1\ \%$$

χ as a function of relative bandwidth, χ(u):

$$\chi(u) := -15.943\cdot u^3 + 21.68\cdot u^2 - 11.707\cdot u + 4.4221 \qquad \text{EQ 29} \qquad < +/-\ 0.5\%$$

Where:

$$u(\chi) = \frac{\omega_{3dBf}(\chi)}{\omega_D} \qquad \text{EQ 30}$$

Therefore $\omega_{3dBf}(\chi) = u(\chi)\cdot\omega_D$ EQ 31

U(χ) vs χ for some selected datapoints ***Actual Values***

$u(\chi) =$	$\chi =$	$\zeta(\chi) =$
0.4995	2	0.5
0.47037	2.05	0.525
0.44297	2.1	0.55
0.41723	2.15	0.575
0.39309	2.2	0.6
0.37047	2.25	0.625
0.34931	2.3	0.65
0.32954	2.35	0.675
0.31109	2.4	0.7
0.29388	2.45	0.725
0.27786	2.5	0.75
0.26295	2.55	0.775
0.24909	2.6	0.8
0.2362	2.65	0.825
0.22422	2.7	0.85
0.21307	2.75	0.875
0.20269	2.8	0.9
0.19302	2.85	0.925
0.18397	2.9	0.95
0.17549	2.95	0.975

$$\frac{\omega_{3dBf}(2)}{\omega_D} = 0.5$$

$$\frac{\omega_{3dBf}(2.25)}{\omega_D} = 0.3702$$

$$\frac{\omega_{3dBf}(2.5)}{\omega_D} = 0.27682$$

$$\frac{\omega_{3dBf}(2.7)}{\omega_D} = 0.22416$$

$$\frac{\omega_{3dBf}(2.75)}{\omega_D} = 0.2134$$

$$\frac{\omega_{3dBf}(2.90)}{\omega_D} = 0.1853$$

$$\frac{\omega_{3dBf}(2.95)}{\omega_D} = 0.17719$$

All Errors are < 1 %

Say we want to find χ that yields a $\omega_{3dBf} = 0.25\ \omega_D$ (for any ω_D)
Using EQ 29

$$\chi(0.25) = 2.601$$

$\chi = 2.6$ should do it.
"Actual value from a detailed solution is 0.24845" which is within 0.62%.

Note that the inverse (EQ 28) yields

$$u(2.600) = 0.249$$

The relative bandwidth values have an error of less than 1 %.

Part 2 - Analysis of velocity closed loop parameters when Prefiltering (outside the loop) **is used.**

Example 1 (No filter)
$\omega_D = 100$ rad/s $\chi = 4$

Actual values.

$$\omega_z = 6.25\,\frac{rad}{s} \qquad \omega_1(4) = 9.5492\,\frac{rad}{s} \qquad \omega_c(4) = 25\,\frac{rad}{s} \qquad \omega_2(4) = 65.4508\,\frac{rad}{s}$$

$$\omega_{3dB} = 38.34\cdot\frac{rad}{s} \qquad t_{r10\_90} = 0.048\cdot s$$

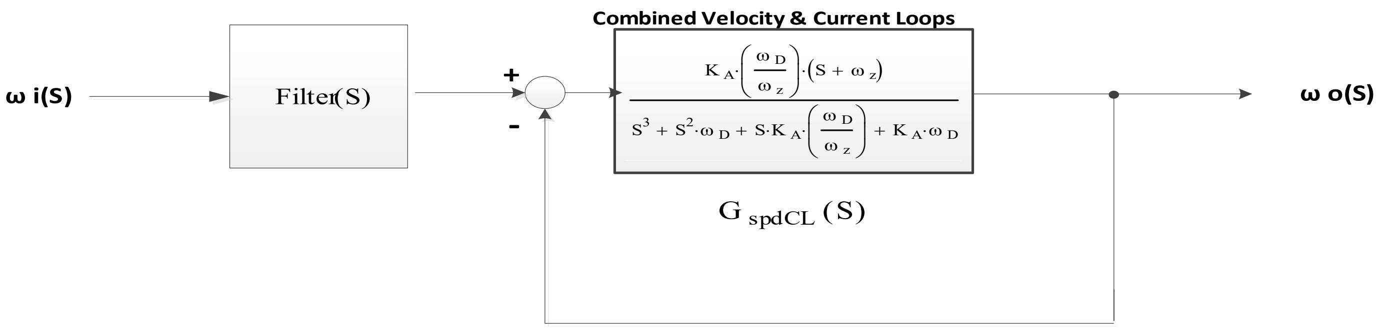
The unfiltered response has significant overshoot (17.3%) in the step response. Also, the closed loop exhibits 1.57 dB of frequency gain peaking. This is caused by the presence of the closed loop zero (ω_z). This is true for any χ, though less pronounced as it is increased in value (e.g., overshoot (OS) = 5.8 %, gain peak = 0.494 dB for $\chi = 13$; overshoot = 3.3 %, gain peak = 0.274 dB for $\chi = 25$). However, if a prefilter is used the overshoot can be reduced to zero, but with a rise time increase and a 3 dB bandwidth decrease. We will show later that this can be a desirable feature.

Refer to Figure 9 (Page 26) for an indication of pole/zero locations.

When X >= 3 ($\zeta >= 1$), as in this example, use a

Lag-Lead Filter $$\text{Filter}(S) = \frac{\omega_z}{\omega_1}\cdot\frac{(S+\omega_1)}{(S+\omega_z)}$$

Here the real pole, $\omega 1$ is the most dominant (except for $\chi = 3$, where all poles are equal).



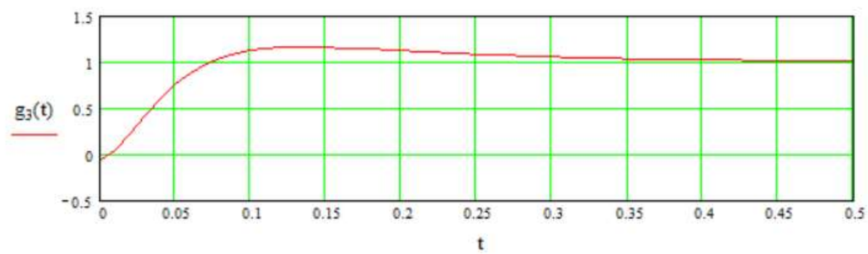
Velocity Loop with External Prefilter
Figure 3

No Prefilter $\chi > 3 , \zeta > 1$

Form 3 $$\frac{1}{S}\cdot\frac{\left(S+\omega_z(\chi)\right)}{\left(S+\omega_1(\chi)\right)\cdot\left(S+\omega_c(\chi)\right)\cdot\left(S+\omega_2(\chi)\right)}$$

Inverse Laplace Transform (Form 3) --

$$g_3(t) := 1 + K_{G1}\cdot\exp\left(-\omega_1(\chi)\cdot t\right) + K_{G2}\cdot\exp\left(-\omega_c(\chi)\cdot t\right) + K_{G3}\cdot\exp\left(-\omega_2(\chi)\cdot t\right) \quad \text{EQ 25}$$



Step Response with a Prefilter

Prefilter $$F(S) = \frac{\omega_z(\chi)}{\omega_1(\chi)}\cdot\frac{\left(S+\omega_1(\chi)\right)}{\left(S+\omega_z(\chi)\right)}$$

Form 3 $$\frac{1}{S}\cdot\frac{\left(S+\omega_z(\chi)\right)}{\left(S+\omega_1(\chi)\right)\cdot\left(S+\omega_c(\chi)\right)\cdot\left(S+\omega_2(\chi)\right)}\cdot\frac{\omega_1(\chi)\cdot\omega_c(\chi)\cdot\omega_2(\chi)}{\omega_z(\chi)}$$

$$s_1 := \left[\zeta_f + \sqrt{\left(\zeta_f\right)^2 - 1}\right]\omega_{nf} \qquad s_2 := \left(\zeta_f - \sqrt{\zeta_f^{\,2} - 1}\right)\cdot\omega_{nf}$$

With a Prefilter the transfer function becomes:

$$\frac{1}{S}\cdot\frac{\omega_{nf}^{\,2}}{\left(S+s_1\right)\cdot\left(S+s_2\right)} \qquad [\zeta > 1]$$

The inverse Laplace Transform is

$$\omega_n^{\,2}\cdot\left[\frac{1}{\left(s_1\cdot s_2\right)} + \frac{1}{\left[s_1\cdot\left(s_1 - s_2\right)\right]}\cdot\exp\left(-s_1\cdot t\right) - \frac{1}{\left[s_2\cdot\left(s_1 - s_2\right)\right]}\cdot\exp\left(-s_2\cdot t\right)\right]$$

$$s_1 - s_2 = 2\cdot\sqrt{\zeta_f^{\,2} - 1}\cdot\omega_{nf} \qquad \omega_{nf}^{\,2} \qquad s_1\cdot s_2 = \omega_{nf}^{\,2}$$

Prefilter

$$\omega_{nf} := \sqrt{\omega_c(\chi)\cdot\omega_2(\chi)} \qquad \omega_{nf} = 40.451\,\text{Hz}$$

$$\zeta_f := \frac{1}{2}\cdot\frac{(\omega_c(\chi)+\omega_2(\chi))}{\omega_{nf}} \qquad \zeta_f = 1.118$$

$$\frac{1}{S}\cdot\frac{\omega_{nf}^{2}}{(S+s_1)\cdot(S+s_2)} \qquad [\zeta > 1]$$

Using the Form 3 Inverse Laplace Transform - See *Inverse Laplace Transform of Common Canonical Forms* EQ 9

The inverse Laplace Transform is

$$\omega_{nf}^{2}\cdot\left[\frac{1}{(s_1\cdot s_2)}+\frac{1}{[s_1\cdot(s_1-s_2)]}\cdot\exp(-s_1\cdot t)-\frac{1}{[s_2\cdot(s_1-s_2)]}\cdot\exp(-s_2\cdot t)\right]$$

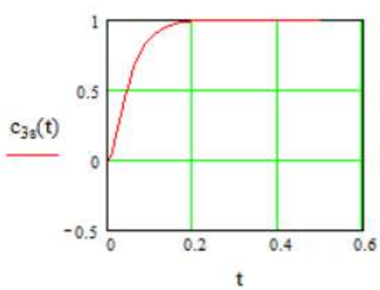
$$s_1 - s_2 = 2\cdot\sqrt{\zeta_f^{2}-1}\cdot\omega_{nf} \qquad \omega_{nf}^{2} \qquad s_1\cdot s_2 = \omega_{nf}^{2}$$

Where -

$$s_1 = 65.451\,\frac{\text{rad}}{\text{s}} \qquad s_2 = 25\,\frac{\text{rad}}{\text{s}}$$

$$\omega_2(\chi) = 65.451\,\frac{\text{rad}}{\text{s}} \qquad \omega_c(\chi) = 25\,\frac{\text{rad}}{\text{s}}$$

$$c_{3s}(t) := \left[1+\frac{\omega_{nf}}{2\cdot\sqrt{\zeta_f^{2}-1}}\cdot\left(\frac{\exp(-s_1\cdot t)}{s_1}-\frac{\exp(-s_2\cdot t)}{s_2}\right)\right] \qquad \text{EQ 9}$$



$$c_{3s}(0.01343\cdot s) = 0.1$$

$$c_{3s}(0.1112s) = 0.9$$

$$t_{r10\_90} := 0.1112\cdot s - 0.01343\cdot s$$

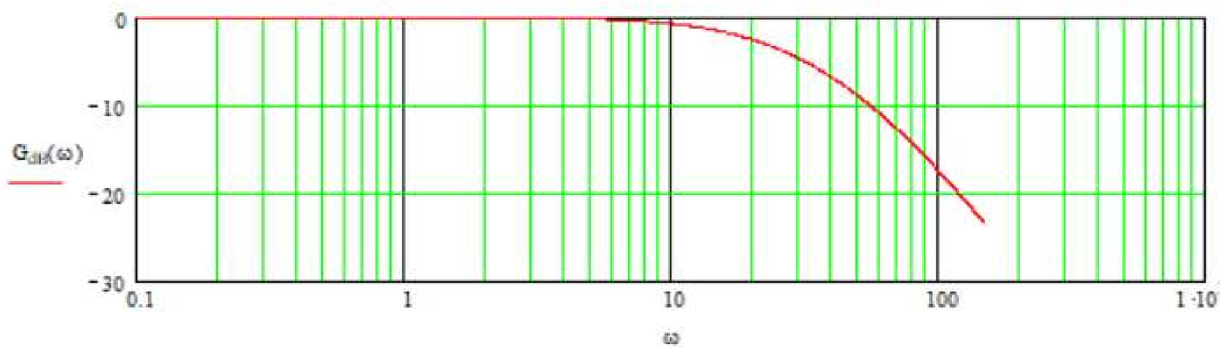
$$t_{r10\_90} = 0.0978\,s$$

No overshoot!

Frequency Response -

$$G(\omega) := \frac{1}{\left(1 + j\cdot\frac{\omega}{\omega_c(\chi)}\right)\left(1 + j\cdot\frac{\omega}{\omega_2(\chi)}\right)}$$

$$G_{dB}(\omega) := 20\cdot\log(|G(\omega)|)$$



No Prefilter

ω3dB = 38.34 rad/s

With Prefilter

ω3dBf = 22.21 rad/s

No gain peaking!

For X < 3 use a simple

Lag Filter $$\text{Filter}(S) = \frac{\omega_z}{(S + \omega_z)}$$

Example 2 with $\chi = 2.716$

$\omega_z(\chi) = 13.556 \frac{rad}{s}$ $\omega_1(\chi) = 31.591 - 18.912i \frac{rad}{s}$ $\omega_c(\chi) = 36.819 \frac{rad}{s}$ $\omega_2(\chi) = 31.591 + 18.912i \frac{rad}{s}$

$\omega_{3dB} = 61.4 \cdot \frac{rad}{s}$ $t_{r10\_90} = 0.029899 \cdot s$

The unfiltered time domain response has 28.5% overshoot in the step response. The closed loop peaking is 2.64 dB @ 24.4 rad/s.

With prefilter

$$\frac{\lambda}{(s+\lambda)} \cdot \frac{\omega_n^{\,2}}{s^2 + 2\zeta \cdot \omega_n \cdot s + \omega_n^{\,2}}$$ **[0< ζ < 1]**

$\alpha = \zeta \cdot \omega_n$ $\omega_d = \omega_n \cdot \sqrt{1 - \zeta^2}$

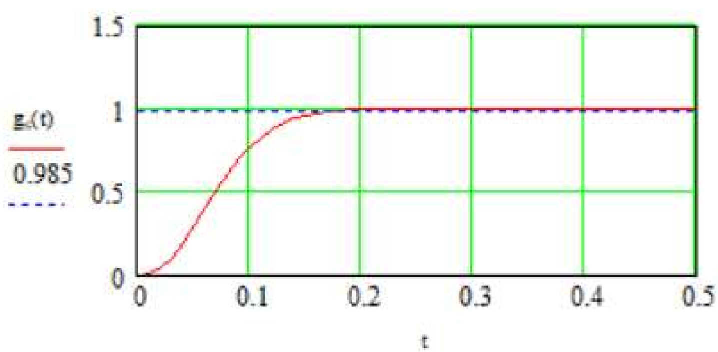
$\lambda := \omega_c(\chi)$ $\zeta = 0.858$

Using the Form 1 Inverse Laplace Transform - See *Inverse Laplace Transform of Common Canonical Forms* EQ 18A, EQ 18B, EQ 18C EQ 19

$\lambda = 36.819 \frac{rad}{s}$ $\alpha = 31.591 \frac{rad}{s}$ $\omega_d = 18.912 \frac{rad}{s}$ $\omega_n = 36.819 \frac{rad}{s}$

$\beta := \frac{\lambda}{\alpha}$ $\beta = 1.166$

$$g_s(t) := 1 - \frac{\exp(-\lambda \cdot t)}{\beta \cdot \zeta^2 \cdot (\beta - 2) + 1} - \frac{\exp(-\alpha \cdot t)}{\beta \cdot \zeta^2 \cdot (\beta - 2) + 1} \cdot \left[\left[\frac{\beta \cdot \zeta \cdot \left[\zeta^2 \cdot (\beta - 2) + 1\right]}{\sqrt{1 - \zeta^2}}\right] \cdot \sin(\omega_d \cdot t) + \beta \cdot \zeta^2 \cdot (\beta - 2) \cdot \cos(\omega_d \cdot t)\right]$$



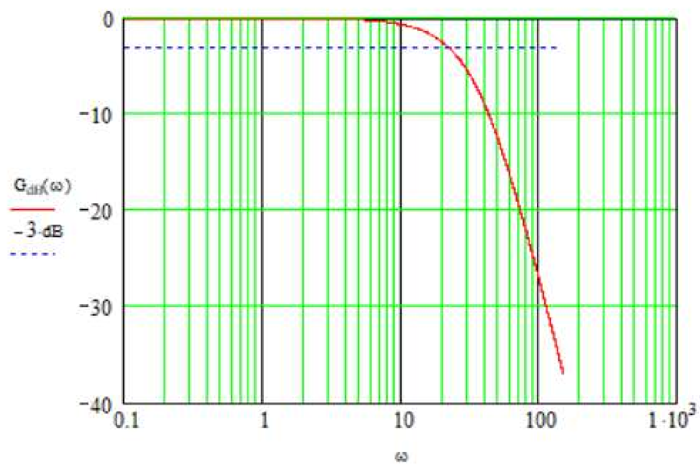
Rise time is 0.097 s, overshoot is 0.064%.

Velocity Closed Loop Response

with prefilter.

$$G(\omega) := \frac{1}{\left(1 + j\cdot\frac{\omega}{\omega_1(\chi)}\right)\cdot\left(1 + j\cdot\frac{\omega}{\omega_2(\chi)}\right)\cdot\left(1 + j\cdot\frac{\omega}{\omega_c(\chi)}\right)}$$

Let $G_{dB}(\omega) := 20\cdot\log(|G(\omega)|)$



$$\omega_{3dBy} := 25\cdot\frac{rad}{s}$$

Given

$$G_{dB}(\omega_{3dBy}) = -3\cdot dB$$

$$\omega_{3dBf} := Find(\omega_{3dBy})$$

$$\omega_{3dBf} = 22.063\,\frac{rad}{s}$$

No Prefilter	With Prefilter
ω3dB = 61.4 rad/s	ω3dBf = 22.06 rad/s

No gain peaking!

Part 3 – Range Adjustment for Limited Span Method.

Changing ω_{3dBf}

Let the desired filtered BW be defimed

$\omega_{3dBf} = u_x \cdot \omega_D$ where $u_x = k \cdot u(\chi)$ k is any positive number

Modify $\omega_{D\_new} = \dfrac{u_x}{u(\chi)} \cdot \omega_D = k \cdot \omega_D$

Therefore $k = \dfrac{u_x}{u(\chi)}$

So $\omega_{D\_new} \cdot u(\chi) = (k \cdot \omega_D) \cdot u(\chi) = u_x \cdot \omega_D$

$\omega_{D\_new} = k \cdot \omega_D$ EQ 32

Map within current Range

Example 1

$u_x := 0.33$ Map to $u(2) = 0.4995$

$k_1 := \dfrac{u_x}{u(2)}$ $k_1 = 0.66066$

$\omega_{D\_new\_1} = k_1 \cdot \omega_D$

$(k_1 \cdot \omega_D) \cdot u(2) = 0.33 \cdot \omega_D$ Checks

Example 2

$u_x := 0.33$ Map to $u(2.75) = 0.21307$

$k_2 := \dfrac{u_x}{u(2.75)}$ $k_2 = 1.549$

$\omega_{D\_new\_2} = k_2 \cdot \omega_D$

$(k_2 \cdot \omega_D) \cdot u(2.75) = 0.33 \cdot \omega_D$ Checks

Map outside current Range

Example 3

$u_x := 0.57$ Map to $u(2) = 0.4995$

$$k_3 := \frac{u_x}{u(2)} \qquad k_3 = 1.14114$$

$$\omega_{D\_new\_3} = k_3 \cdot \omega_D$$

$$(k_3 \cdot \omega_D) \cdot u(2) = 0.57 \cdot \omega_D$$ Checks

Example 4

$u_x := 0.062$ Map to $u_{2\_95} = 0.17719$ use actual value

$$k_4 := \frac{u_x}{u_{2\_95}} \qquad k_4 = 0.34991$$

$$\omega_{D\_new\_4} = k_4 \cdot \omega_D$$

$$(k_4 \cdot \omega_D) \cdot u_{2\_95} = 0.062 \cdot \omega_D$$ Checks

-

Note:

If we used the uncompensated u(2.95) then $u(2.95) = 0.17549$

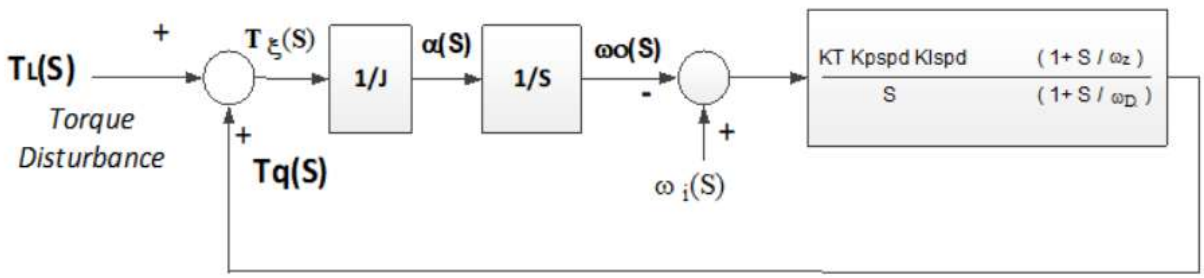
$$k_{4x} := \frac{u_x}{u(2.95)} \qquad k_{4x} = 0.3533$$

$$(k_{4x} \cdot \omega_D) \cdot u(2.95) \quad = \quad 0.34991 \cdot u(2.95) \cdot \omega_D = 0.062 \cdot \omega_D$$

Actually a + 0.97 % error in u_x $\quad 0.3533 \cdot u_{2\_95} \cdot \omega_D = 0.0626 \cdot \omega_D$

Part 4 - Torque Disturbance Impact and Comparison of Design Methods.

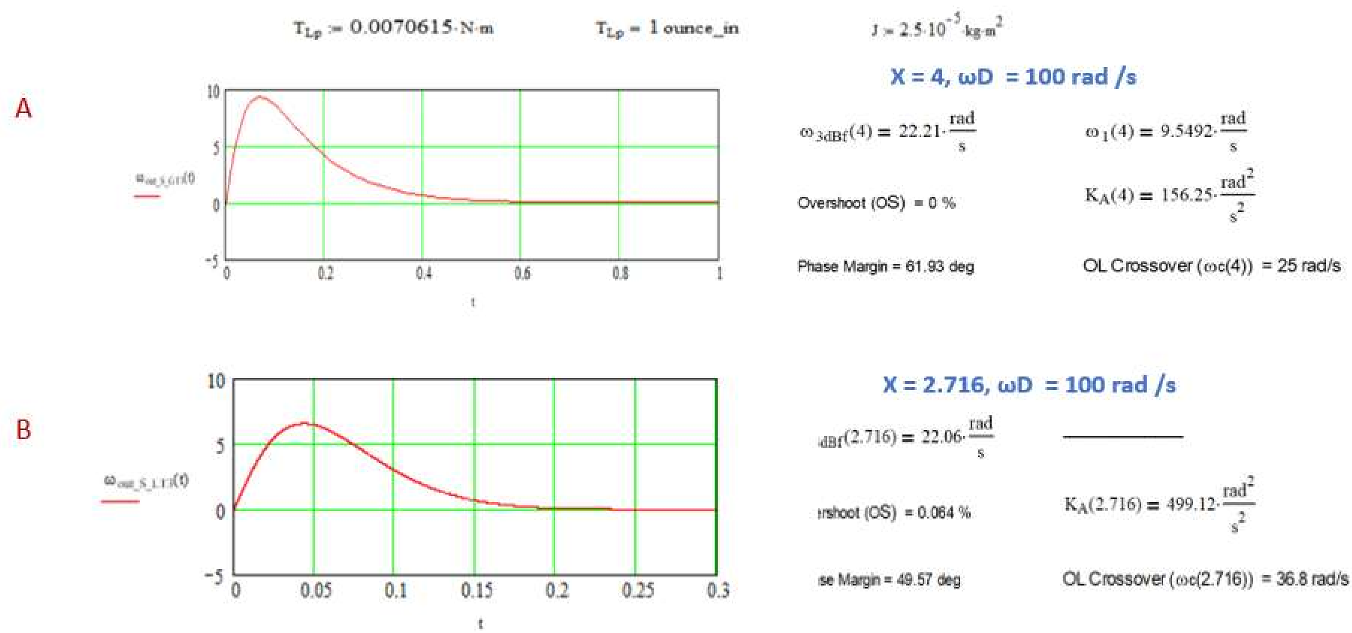
Note that there is NO prefilter influence due to the fact that it is outside the velocity loop.
Refer to the document *Cascade PI Loop Torque Disturbance Analysis*, by this author



Torque Disturbance Model
Figure 5

Step of Torque Response – With *approximately (< 0.7 %) equal bandwidths by using E*Q 29 and EQ 30 for B. Refer to Figures 6A and 6B below.

$T_{Lp} := 0.0070615 \cdot N \cdot m$ **Step Input Amplitude** $J = 2.5 \times 10^{-5}\,kg\,m^2$



Figures 6A and 6B

For A the disturbance peak is -9.313 rad / s @ 0.06981 s, 1% of peak recovery time ~ 0.603 s (**$5.2/\omega_1(4)$** = 0.5446 s).
For B the disturbance peak is -6.511 rad / s @ 0.0432 s, 1% of peak recovery time ~ 0.200 s (**$12.5/\omega_{3dB}(2.716)$** ~ 0.202 s).
Note $\omega_{3dB}(\chi)$ is found from EQ 23 which is 1.68 (ω_D / χ).

Note the slower recovery time of (A) due to the dominant real pole ω_1, which was created because χ > 3.

For (B), as $\chi < 3$, $\omega_1(\chi)$ and $\omega_2(\chi)$ are a complex conjugate pair while $\omega_c(\chi)$ is real, and the response is faster due to ω_{3dB} dependance.

Define the time that ω_{out} settles to within 1% of the disturbance peak (residual) as $T_{\Delta\omega out\_peak*0.99}(\chi)$.

Empirically what was found -- For $\chi < 3$, $T_{\Delta\omega out\_peak*0.99}(\chi) \sim 12.5 / \omega_{3dB}(\chi)$ (+/- 20%);
For $\chi > 4$, $T_{\Delta\omega out\_peak*0.99}(\chi) \sim 5.2 / \omega_1(\chi)$ (+/- 10 %)

1% of peak recovery time w*hen $\chi = 3$, the best approximation is $28.06 / \omega_D$.*

Useful Equations for a step of torque disturbance –

Time estimates for the occurrence of the peak:

For 2 <= χ < 4

Error < +/- 3.0 %

$$T_{peak\_L}(\chi, \omega_D) := \frac{\left(0.2163 \cdot \chi^2 + 0.7558 \cdot \chi + 0.6984\right)}{\omega_D} \qquad \text{EQ 33}$$

For 4 <= χ <=24.5

Error < +/- 3.0 %

$$T_{peak\_U}(\chi, \omega_D) := \frac{0.0409 \cdot \chi^2 + 2.4526\chi - 3.685}{\omega_D} \qquad \text{EQ 34}$$

From the above example -

$$T_{peak\_L}\left(2.716, 100 \cdot \frac{rad}{s}\right) = 0.043\,s \qquad T_{peak\_U}\left(4, 100 \cdot \frac{rad}{s}\right) = 0.068\,s$$

Peak of offset radian frequency estimates:

For 2 <= χ < 4

Let

$$a := -0.35887268 \qquad \omega_{ref\_L} := 0.74967119 \cdot \frac{rad}{s}$$

Then

Error < +/- 1.0 %

$$\Delta\omega_{o\_peak\_L}(\chi) := \left[(\chi - a) \cdot \left(\frac{1 \cdot \frac{rad}{s}}{\omega_D} \cdot \frac{T_{Lp}}{1 \cdot N \cdot m} \cdot \frac{1 \cdot kg \cdot m^2}{J}\right)\right] \cdot \omega_{ref\_L} \qquad \text{EQ 35}$$

For 4 <= χ <=24.5

Let

$$b := 0.46092839 \qquad \omega_{ref\_U} := 0.91066589 \cdot \frac{rad}{s}$$

Then

Error < +/- 2.5 %

$$\Delta\omega_{o\_peak\_U}(\chi) := \left[(\chi - b)\cdot\left(\frac{1\cdot\frac{rad}{s}}{\omega_D}\cdot\frac{T_{Lp}}{1\cdot N\cdot m}\cdot\frac{1\cdot kg\cdot m^2}{J}\right)\right]\cdot\omega_{ref\_U} \qquad \text{EQ 36}$$

With ω_D = 100 rad/s, T_{Lp} = 0.0070615 Nm (1 oz-in) and J = 2.5 x 10^{-5} kg m^2.
From the above example -

$$\Delta\omega_{o\_peak\_L}(2.716) = 6.511\cdot\frac{rad}{s} \qquad \Delta\omega_{o\_peak\_U}(4.00) = 9.103\cdot\frac{rad}{s}$$

The attenuation improvement or degradation (disturbance rejection ratio) is determined by evaluating

$$\frac{\Delta\omega_{o\_peak\_i}(\chi_i)}{\Delta\omega_{o\_peak\_j}(\chi_j)} \qquad \text{where i and j = L or U}$$

When $\left|\Delta\omega_{o\_peak\_i}(\chi_i)\right| < \left|\Delta\omega_{o\_peak\_j}(\chi_j)\right|$ there is greater attenuation for condition i then for condition j – Under this condition, i has the better disturbance rejection.

For a sinusoidal load disturbance, the transfer function is (Refer to the document *Cascade PI Loop Torque Disturbance Analysis*):

$$\frac{T_\xi(S)}{T_L(S)} = \frac{S^2\cdot(S+\omega_D)}{S^3 + S^2\cdot\omega_D + S\cdot\frac{\omega_D^{\,2}}{\chi} + \frac{\omega_D^{\,3}}{\chi^3}} \qquad \text{EQ 37}$$

$$S = j\cdot\omega$$

$$G_\xi(\omega) = \frac{T_\xi(\omega)}{T_L(\omega)} \quad \text{EQ 38} \qquad \tau_\xi(\omega_{test}) = \frac{-\phi_\xi(\omega_{test})}{\omega_{test}} \quad \text{EQ 39}$$

With

$$T_L(t) := T_{Lp}\cdot\cos(\omega_{test}\cdot t)$$

At the torque error point -

$$T_\xi(t) = T_L(t - \tau_\xi(\omega_{test}))\cdot\left|G_\xi(\omega_{test})\right| \qquad \text{EQ 40}$$

Now

$$\omega_{out}(t) = \frac{1}{J}\cdot\int T_\xi(t)\,dt \qquad \text{EQ 41}$$

The disturbance is modified at the frequency ω_{test} by:
Magnitude -

$$T_P = T_{Lp} \cdot \left|G_\xi(\omega_{test})\right| \qquad \textbf{EQ 42}$$

Phase -

$$\Theta = \arg(G_\xi(\omega_{test})) \qquad \textbf{EQ 43}$$

. So

$$T_\xi(t) = T_P \cdot \cos(\omega_{test} \cdot t + \Theta) \qquad \textbf{EQ 44}$$

and

$$\omega_{out}(t) = \frac{1}{J} \cdot \frac{T_P}{\omega_{test}} \cdot \sin(\omega_{test} \cdot t + \Theta) \qquad \textbf{EQ 45}$$

The Sensitivity Function at any frequency ω_{test} is given by:

$$S_\xi(\omega) = \frac{1}{1 + G_{ol}(\omega)} \qquad \textbf{EQ 46}$$

$$S_{\xi dB} = 20 \cdot \log(\left|S_\xi(\omega)\right|) \qquad \textbf{EQ 47}$$

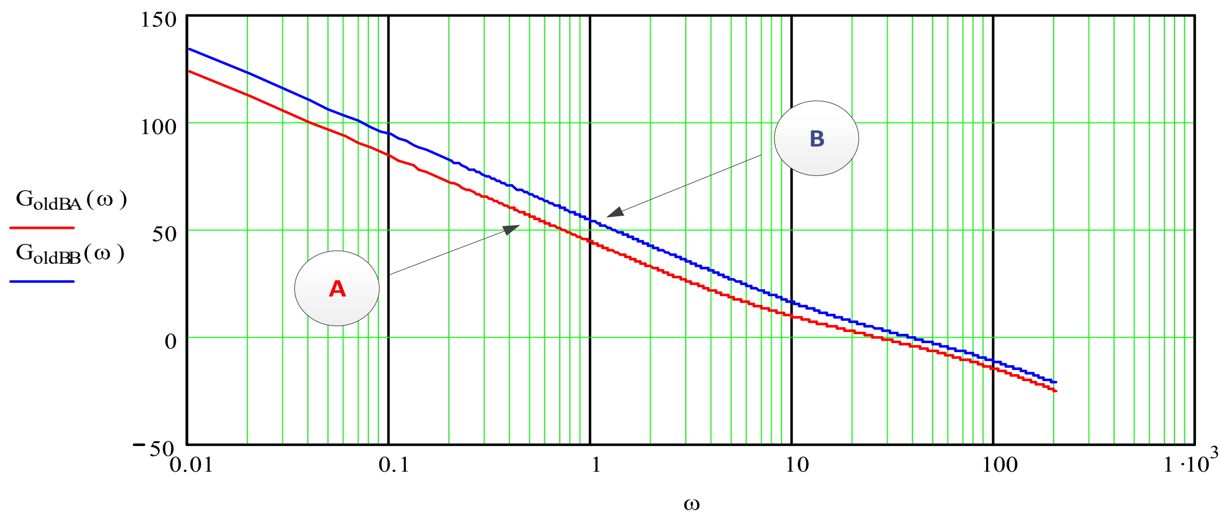
From EQ 9, the open loop gain is proportional to K_A. Therefore, the larger the K_A the smaller the Sensitivity Function's value (and the better the disturbance attenuation).

A	**B**
$\chi = 4$	$\chi = 2.716$
$\omega_{3dBA} := 22.21 \cdot \frac{rad}{s}$	$\omega_{3dBB} := 22.06 \cdot \frac{rad}{s}$
$K_{AA} := 156.25 \cdot \frac{rad^2}{s^2}$	$K_{AB} := 499.127 \cdot \frac{rad^2}{s^2}$
$\omega_{zA} := 6.25 \cdot \frac{rad}{s}$	$\omega_{zB} := 13.556 \cdot \frac{rad}{s}$
Overshoot $= 0 \cdot \%$	Overshoot $= 0.064\%$

$$\omega_D := 100 \cdot \frac{rad}{s}$$

$$G_{olA}(\omega) := \frac{-K_{AA}}{\omega^2} \cdot \frac{\left(1 + j \cdot \frac{\omega}{\omega_{zA}}\right)}{\left(1 + j \cdot \frac{\omega}{\omega_D}\right)} \qquad G_{olB}(\omega) := \frac{-K_{AB}}{\omega^2} \cdot \frac{\left(1 + j \cdot \frac{\omega}{\omega_{zB}}\right)}{\left(1 + j \cdot \frac{\omega}{\omega_D}\right)}$$

$$G_{oldBA}(\omega) := 20 \cdot \log\left(\left|G_{olA}(\omega)\right|\right) \qquad G_{oldBB}(\omega) := 20 \cdot \log\left(\left|G_{olB}(\omega)\right|\right)$$



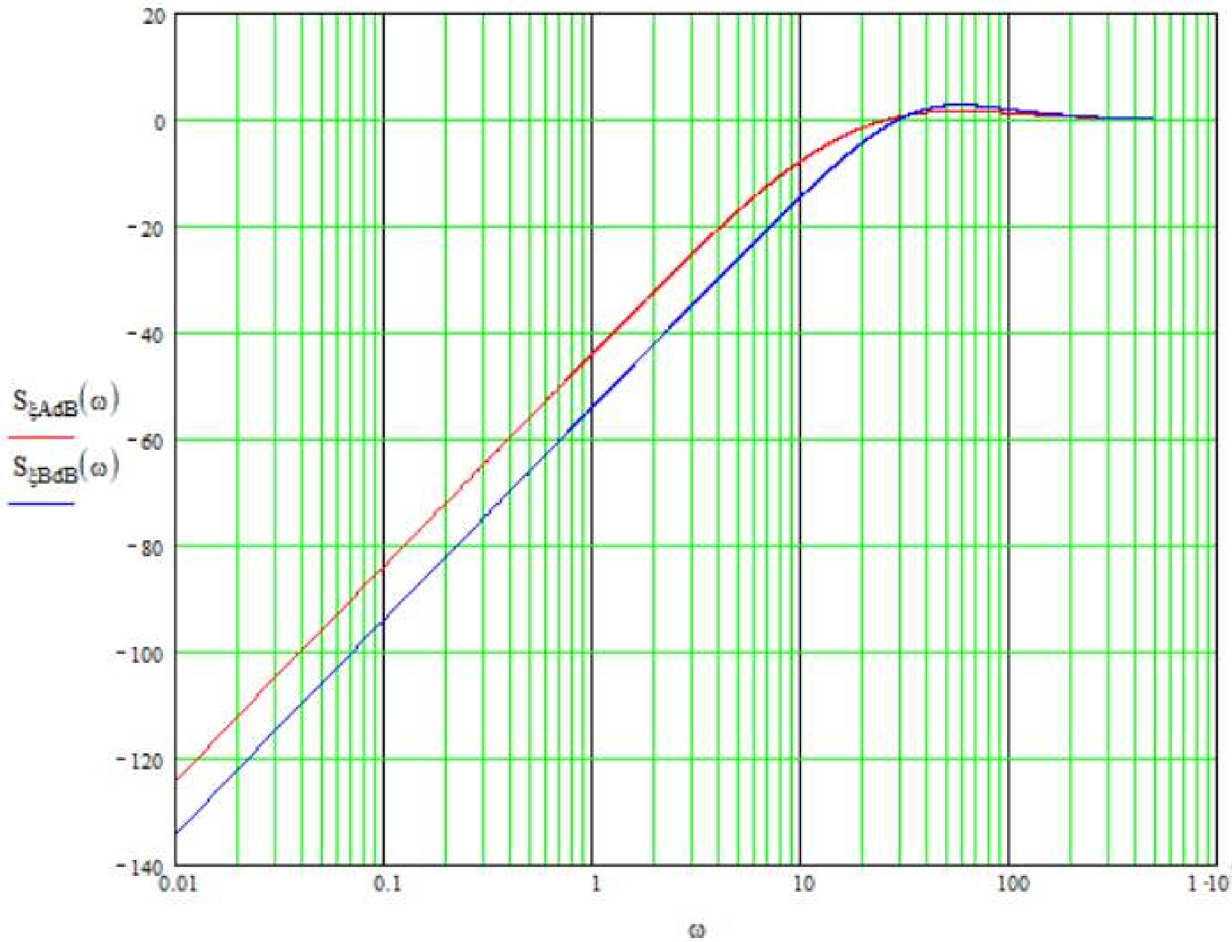
Open Loop Gain Comparisons
With ~ the same Prefiltered Bandwidth

Figure 7

Sensitivity Functions are given by:

$$S_{\xi A}(\omega) := \frac{\frac{(j \cdot \omega)^2}{K_{AA}} \cdot \left(1 + j \cdot \frac{\omega}{\omega_D}\right)}{(j \cdot \omega)^3 \cdot \frac{\chi_A{}^3}{\omega_D{}^3} + (j \cdot \omega)^2 \cdot \frac{\chi_A{}^3}{\omega_D{}^2} + (j \cdot \omega) \cdot \frac{\chi_A{}^2}{\omega_D} + 1}$$

$$S_{\xi B}(\omega) := \frac{\frac{(j \cdot \omega)^2}{K_{AB}} \cdot \left(1 + j \cdot \frac{\omega}{\omega_D}\right)}{(j \cdot \omega)^3 \cdot \frac{\chi_B{}^3}{\omega_D{}^3} + (j \cdot \omega)^2 \cdot \frac{\chi_B{}^3}{\omega_D{}^2} + (j \cdot \omega) \cdot \frac{\chi_B{}^2}{\omega_D} + 1}$$



Sensitivity Function Comparisons
With ~ the same Prefiltered Bandwidth

Figure 8

$$20 \cdot \log\left(\frac{K_{AB}}{K_{AA}}\right) = 10.088 \cdot dB$$

Relative Span Method Sensitivity Function Improvement @ low frequencies ($\omega << \omega_{zA}$)

Technically a –10.088 dB (disturbance attenuation of B relative to A).

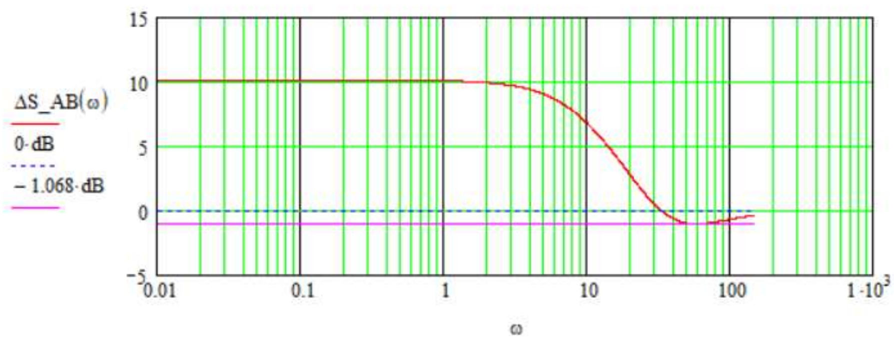
For larger frequencies ($\omega > \omega_{zB}$) The roles will eventually reverse. In this example the crossover frequency is ω_{cross} = 32.78 rad /s.

MAX $S_{\xi AdB}\left(57.329 \cdot \frac{rad}{s}\right) = 1.568\,dB$ $S_{\xi BdB}\left(57.80 \cdot \frac{rad}{s}\right) = 2.6365\,dB$

Crossover $S_{\xi AdB}\left(32.7804 \cdot \frac{rad}{s}\right) = 0.8081\,dB$ $S_{\xi BdB}\left(32.7804 \cdot \frac{rad}{s}\right) = 0.8081\,dB$

$S_{\xi AdB}\left(100 \cdot \frac{rad}{s}\right) = 1.1591\,dB$ $S_{\xi BdB}\left(100 \cdot \frac{rad}{s}\right) = 1.8647$

Let $$\Delta S\_AB(\omega) := S_{\xi AdB}(\omega) - S_{\xi BdB}(\omega)$$



Maximum difference is -1.068 dB in this case.
Asymptotically approaches 0 dB as ω increases beyond ω_{max}.
Remember that the best performance (lowest Max difference) is achieved when identical ω_D's are compared with the same ω_{3dBf}.

Importance of K_A [Bigger is better].

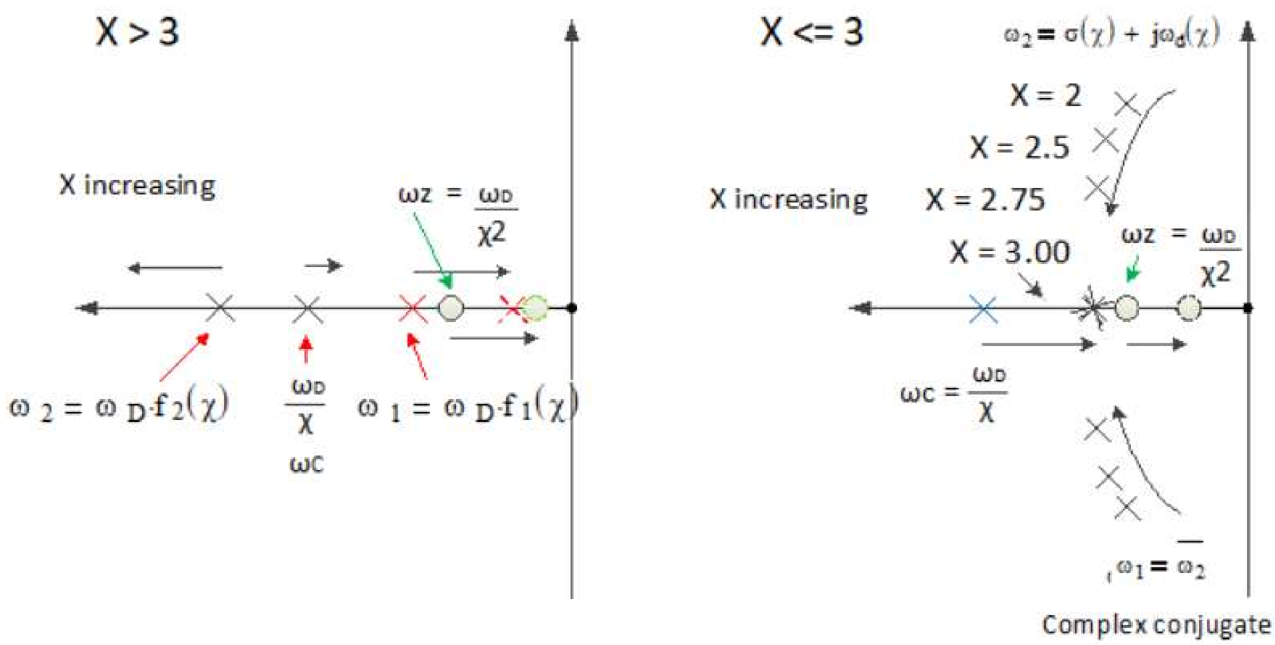
Useful equations are:

$$G_{oldB}(\omega) = 40 \cdot \log\left(\frac{\sqrt{K_A}}{\omega}\right)$$ for $\omega <= \omega_z$

$$\omega_o = \omega_z \cdot \chi = \frac{\omega_D}{\chi} = \omega_c$$

open loop 0 dB
crossover frequency calculation

The Closed Loop Root Locus as a function of χ is shown in Figure 9 for clarity



Closed Loop Pole-Zero Positions as a Function of χ

Figure 9

Note:

$$f_1(\chi) = \frac{1}{2\cdot\chi}\cdot\left(-1 + \chi - \sqrt{\chi^2 - 2\chi - 3}\right) \qquad \text{From EQ 15}$$

$$f_2(\chi) = \frac{1}{2\cdot\chi}\cdot\left(-1 + \chi + \sqrt{\chi^2 - 2\chi - 3}\right) \qquad \text{From EQ 16}$$

Interestingly enough, the ω_{3dBf} values obtained for the range $3 < \chi < 5.35$ are identical to the values obtained using $2 < \chi < 2.95$. Above χ equals 3 the dominate pole is real and designated by ω_1, as compared to ω_{3dB} for the $\chi < 3$ case. Hence, the slower disturbance recovery response (1% $\tau_{peak} \sim 5/\omega_1$). Larger K_A means lower closed loop and disturbance errors.

Summary Chart – Overshoot Comparison

Limited Span Symmetry Method / Lag Prefilter – Filter Pole at ω_Z

Limited Span	Span	Overshoot Limited Span
u(χ)	**χ**	**OS**
0.50000	2.000	8.15%
0.38520	2.217	3.86%
0.29775	2.435	1.15%
0.29246	2.451	1.05%
0.23347	2.660	0.24%
0.22060	2.716	0.064%
0.20137	2.810	0.0075%
0.18732	2.888	0%
0.17719	2.950	0%

Table 2

u(χ) are the actual computed values.

Note that the location of the closed loop pole at $\omega_c = \omega_D / \chi$ significantly reduces the overshoot.

The approximate 10 to 90 % rise time (error < 5%) can be found from:

$$t_{r10\_90} = \frac{2.1972}{u(\chi)} \cdot \frac{1}{\omega_D} \qquad \textbf{EQ 48}$$

Part 6 – Conclusions

Standard Symmetry Method (unrestricted χ values) has certain issues:

- When $\chi > 3$, ω_1 is a real pole and will dominate any transient disturbance response See Figures 6A, 6B and Figure 9.

  Avoid use of $\chi > 4$ if possible. A gear box may solve some torque disturbance issues, but test signals applied to the velocity loop will be influenced. Also, a Lag-Lead filter is required when $\chi >= 3$. Position Loops (not covered herein) may be significantly impacted because of the dominance of ω_1 for disturbances. *Only a lag filter is needed for $\chi < 3$.*

- All bandwidth specifications from $3 < \chi < 5.35$ can be replicated with the Limited Span Method with $2.6 <= \chi < 2.95$.

The Limited Span Method always provides a K_A which is larger than that of the Standard Method for an identical ω_{3dBf} regardless of the χ and/or new ω_D ($2 <= \chi <= 2.95$) used**.** It also yields a good overshoot response. Therefore, closed loop errors and disturbance rejection is better. Further, the approximation for the ω_{3dBf} values ($2 <= \chi <= 2.95$), when using EQ 28, EQ 29, EQ 30 and EQ 31) are fairly precise.

Out of band Sensitivity will be only be slightly increased (< 1.5 dB max; for the same ω_D and ω_{3dBf}) for the Limited Span as compared with the Standard Method (when $\chi >= 3$).

As always, the design specifications must be consulted to select the best approach.

One last note. The range adjustment presented in Part 3 can be expanded to include ω_D adjustment for either overshoot or open loop gain and overshoot.

For example, if it is desired to have a step response overshoot of 1.15 % at $\omega_{3dBf}(\chi) = 0.4\ \omega_D$ then obtain ω_{D_new} as follows:

Select u(2.435) = 0.29775 (Table 2) as it provides 1.15 % overshoot.
Set $(k\ \omega_D)\ 0.29775 = 0.4\ \omega_D$, solving for $k = 0.4/0.29775 = 1.3434$
Therefore, $k\ \omega_D = 1.3434\ \omega_D$, this assumes that this increase in ω_D is possible.

For a K_A adjustment (Limited Span Symmetry). use EQ 27 and Table 2.
If we desire $K_A = 600\ rad^2/s^2$ with a 0.24 % overshoot ($\chi = 2.66$).

$K_A(2.660) = \omega_{D\_new}^2 / 2.66^3 = 600\ rad^2/s^2$; $\omega_{D\_new} = 106.27$ rad/s.
$\omega_{3dBf}(2.660) = (0.23347\ \omega_{D\_new}) = 24.81$ rad/s. $K_A = 600\ rad^2/s$.

The following references were found to be very useful.
Note that in the literature β equals χ^2 (Dave Wilson's δ^2).

References:

[1] Wilson Dave, (2015), *Teaching Your PI Controller to Behave Parts 1 through 10.* Texas Instruments. [Teaching Your PI Controller to Behave (Part 1) - Industrial - Technical articles - TI E2E support forums]()

[2] Kessler C, *Das Symmetrische Optimum* (1958) Regelungstetechnik, 6, n° 11 pp 395-400, n° 12 432-436.

[3] I.D Landau and A. Voda, *THE "SYMMETRISCHE OPTIUM" AND THE AUTO-CALIBRATION OF PID CONTROLLERS (1992)*, pp 407-412.

[4] Jeffrey W. Umland and Mohammed Safiuddin (1990), *Magnitude and Symmetric Optimum Criterion for the Design of Linear Control Systems: What Is It and How Does it Compare with Others*. IEEE Transactions On Industry Applications Vol. 26.. No 3 May / June 1990 pp 489-497

[5] Voda, A.A and I.D Landau (1995), *A method for auto-calibration of PID controllers, Automatica,* p 31 and pp 41-53.

[6] Preitl, S and R-E Precup (1999), *An extension of tuning relations after Symmetrical Optimum method for PI and PID controllers*, *Automatica,* p 35 and pp 1731-1736.

[7] Martin Machaba and Martin Braae (2003), Explicit Damping Factor Specification in Symmetrical Optimum *Tuning of PI Controllers,* pp 1-6.

[8] Zsuzsa Preitl, Ruth Bars (2005), *A Youla-parameterization Approach for Controller Design based on ESO and 2E-SO Methods for Low Order Benchmarks.* pp 1-13.

[9] Viorel Nicolau, Constantin Miholca, Dorel Aiordachioaie, Emil Ceanga, (2005), *Aspects of Pole Placement Technique In Symmetrical Optimum Method for PID Controller Design,* pp 1-6.