Cascade Position Loop Analysis when using the Limited Span Method (LSM)

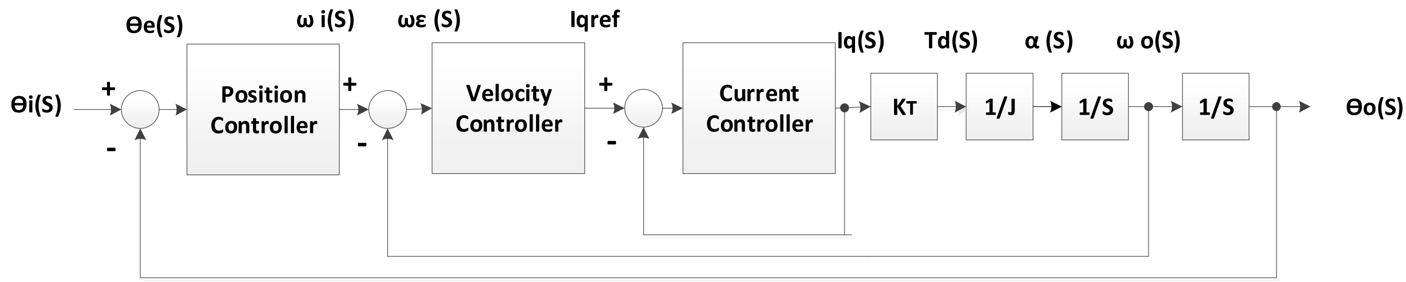


Figure 1A

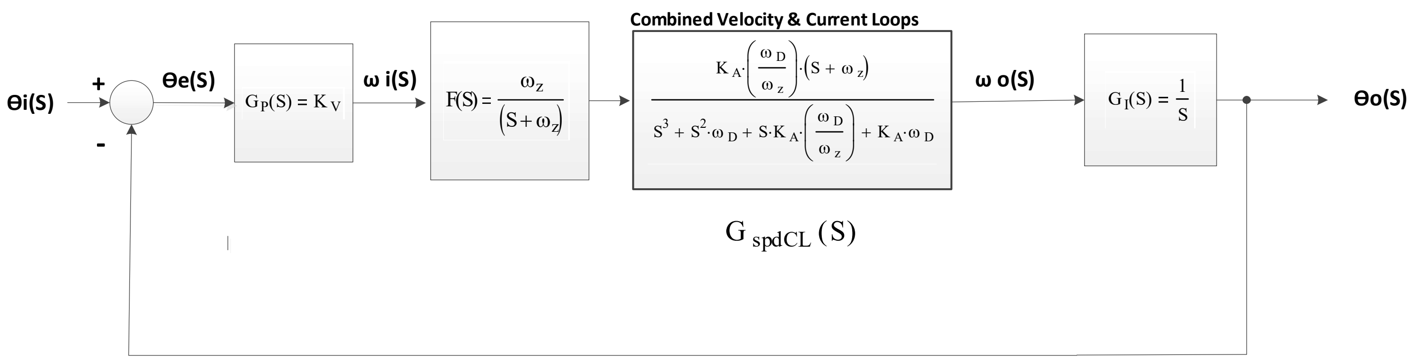


Figure 1B

Multiloop Position Control Models

Provided in Figure 1 is a simplified Block Diagram of a Multiloop Position Control Servo System. Note that the position controller is of a proportional type (with a gain of $K_v\,sec^{-1}$).

With Position Controller Transfer Function

$$G_P(S) = K_v \qquad \text{EQ 1}$$

$K_v\left(s^{-1}\right)$ Units

and Integrator to obtain position from velocity

$$G_I(S) = \frac{1}{S} \qquad \text{EQ 2}$$

From $\omega(t) = \frac{d}{dt}\theta(t)$ so $\theta(t) = \int_0^t \omega(t)\,dt$

If the Laplace Transform of f(t) is F(S) $F(S) = L(f(t))$

Then $\int_0^t f(\tau)\,d\tau$ -------> $\frac{F(S)}{S}$

Static Velocity Error Constant is found from

$$\lim_{S \to 0} S \cdot G_{posOL}(S) = K_v$$

In the literature Kv is used for this constant.
Hence that name is used here.

If the Input is $\theta(t) = \theta_x \cdot t$ Units $\theta_x \left(\frac{rad}{s}\right)$ Slope

Error Value for a input ramp of phase. $e(\infty) = \lim_{t \to \infty} e(t) = \frac{\theta_x}{K_v}$ (rad)

Derivations:

Refer to the document: PI Controller Design using a Limited Symmetrical Method EQ's 3-13.

Closed Loop Velocity / Current Loop

Closed Loop with Prefilter

$$\frac{\omega_o(S)}{\omega_i(S)} = G_{spdCL}(S) = \frac{K_A \cdot \omega_D}{S^3 + S^2 \cdot \omega_D + S \cdot K_A \cdot \frac{\omega_D}{\omega_z} + K_A \cdot \omega_D} \qquad \text{EQ 3}$$

For 2 <= χ <= 2.95 where χ is the span. ζ = 0.5 χ -0.5.

$$\sigma(\chi) = \frac{\omega_D}{2 \cdot \chi} \cdot (\chi - 1) \qquad \text{EQ 4}$$

$$\omega_1(\chi) = \sigma(\chi) \cdot \left[1 - j \cdot \frac{\sqrt{-\chi^2 + 2 \cdot \chi + 3}}{(\chi - 1)} \right] \qquad \text{EQ 5}$$

$$\omega_2(\chi) = \sigma(\chi) \cdot \left[1 + j \cdot \frac{\sqrt{-\chi^2 + 2 \cdot \chi + 3}}{(\chi - 1)} \right] \qquad \text{EQ 6}$$

$$\omega_c(\chi) = \frac{\omega_D}{\chi} \qquad \text{EQ 7}$$

The damping frequency is given by:

$$\omega_d(\chi) = \sigma(\chi) \cdot \frac{\sqrt{-\chi^2 + 2\cdot\chi + 3}}{(\chi - 1)} \qquad \text{EQ 8}$$

Position Loop

Open Loop

$$G_{posOL}(S) = \frac{\frac{K_v}{S} \cdot K_A \cdot \omega_D}{S^3 + S^2 \cdot \omega_D + S \cdot K_A \cdot \frac{\omega_D}{\omega_z} + K_A \cdot \omega_D} \qquad \text{EQ 9}$$

or

$$G_{posOL}(s) = \frac{\frac{K_v}{S} \cdot \frac{\omega_D^3}{\chi^3}}{S^3 + S^2 \cdot \omega_D + S \cdot \frac{\omega_D^2}{\chi} + \frac{\omega_D^3}{\chi^3}} \qquad \text{EQ 10}$$

$S = j \cdot \omega$

$$G_{posOL}(\omega) := \frac{\frac{K_v \cdot \left(\frac{\omega_D^3}{\chi^3}\right)}{j \cdot \omega}}{(j \cdot \omega)^3 + (j \cdot \omega)^2 \cdot \omega_D + (j \cdot \omega) \cdot \frac{\omega_D^2}{\chi} + \frac{\omega_D^3}{\chi^3}} \qquad \text{EQ 11}$$

Magnitude

$$G_{posOLdB}(\omega) = 20 \cdot \log\left(\left|G_{posOL}(\omega)\right|\right) \qquad \text{EQ 12}$$

Phase

$$\phi_{posOL}(\omega) = \arg\left(G_{posOL}(\omega)\right) \qquad \text{EQ 13}$$

Closed Loop

$$G_{posCL}(S) = \frac{K_v \cdot K_A \cdot \omega_D}{S^4 + S^3 \cdot \omega_D + S^2 \cdot K_A \cdot \frac{\omega_D}{\omega_z} + S \cdot K_A \cdot \omega_D + K_v \cdot K_A \cdot \omega_D} \qquad \text{EQ 14}$$

or

$$G_{posCL}(S) = \frac{K_v \cdot \frac{\omega_D{}^3}{\chi^3}}{S^4 + S^3 \cdot \omega_D + S^2 \cdot \frac{\omega_D{}^2}{\chi} + S \cdot \frac{\omega_D{}^3}{\chi^3} + K_v \cdot \frac{\omega_D{}^3}{\chi^3}} \qquad \text{EQ 15}$$

$S = j \cdot \omega$

$$G_{posCL}(\omega) = \frac{K_v \cdot \frac{\omega_D{}^3}{\chi^3}}{(j \cdot \omega)^4 + (j \cdot \omega)^3 \cdot \omega_D + (j \cdot \omega)^2 \cdot \frac{\omega_D{}^2}{\chi} + (j \cdot \omega) \cdot \frac{\omega_D{}^3}{\chi^3} + K_v \cdot \frac{\omega_D{}^3}{\chi^3}} \qquad \text{EQ 16}$$

How to Set Kv?

In the document *PI Controller Design using a Limited Span Symmetrical Method (Part 4)* it was pointed out that when a Limited Span Symmetrical Method ($2 <= \chi < 2.95$) is employed the 3 dB bandwidth is denoted by $\omega_{3dBf}(\chi)$ the filtered Velocity Loop bandwidth. See EQs 28,29,30 and 31 in the referenced document.

Now the Position Loop must have a smaller bandwidth than the Velocity Loop to ensure stability.

$$POS\_BW(\chi) \quad << \quad \omega_{3dBf}(\chi) \qquad \text{EQ 17}$$

Typically, the position bandwidth is from 5 to 10 times smaller. Therefore, the maximum 3dB Position Closed Loop Bandwidth (POS_BW) is:

$$POS\_BW(\chi, M) = \frac{\omega_{3dBf}(\chi)}{M} \qquad \text{EQ 18}$$

where $5 \le M \le 10$ M = 5 optimistic

M = 10 conservative

Kv for a position bandwidth which is approximately 1 /10 of the velocity loop bandwidth (ω_{3dBf}) can be found using equation 19 below. The accuracy here is within +/- 0.3 %. Refer to Table 1.

$$K_v(\chi, M = 10) = \frac{\omega_{3dBf}}{10} \cdot (0.04432 \chi + 0.73417) \qquad \text{EQ 19}$$

This is valid over the entire Limited Span Range specified above.

Kv for a position bandwidth which is approximately 1 /5 of the velocity loop bandwidth (ω_{3dBf}) can be found using equation 20 below. The accuracy here is within +/- 0.65 %. Refer to Table 2.

$$K_v(\chi, M = 5) = \frac{\omega_{3dBf}}{5} \cdot (0.077 \cdot \chi + 0.5311) \qquad \text{EQ 20}$$

This is valid over the entire Limited Span Range specified above.

As the characteristic equation is of fourth order, which for a limited span system, yields two real poles and one complex conjugate pole pair, the exact form for the step response is given as:

$$s_1 = \sigma_1 \qquad s_2 = \sigma_2 \qquad s_3 = \sigma_3 - j \cdot \omega_3 \qquad s_4 = \sigma_3 + j \cdot \omega_3$$

$$\text{With} \quad \omega_n = \sqrt{\sigma_3^{\,2} + \omega_3^{\,2}} \qquad \zeta = \frac{\sigma_3}{\omega_n} \qquad \omega_d = \omega_3 = \omega_n \cdot \sqrt{1 - \zeta^2}$$

Let θ_s represent the amplitude of a step of phase, in radians.
The output phase response is given by -

$$\theta_{Ostep}(t) = \theta_s \cdot \left[1 + a_1 \cdot \exp(-\sigma_1 \cdot t) + a_2 \cdot \exp(-\sigma_2 \cdot t) + \exp(-\sigma_3 \cdot t) \cdot \left(b_1 \cdot \cos(\omega_3 \cdot t) + b_2 \cdot \sin(\omega_3 \cdot t)\right) \right] \qquad \text{EQ 21}$$

A very good approximation to EQ 21 (< 1.5 % rise time difference) is given by:

$$\theta_{Ostep}(t) \sim \theta_s \left[1 - \exp\left[\frac{-\omega_{3dBf}}{M} \cdot (t - \Delta) \right] \right] \cdot \Phi(t - \Delta) \qquad \text{EQ 22}$$

$\Phi(x)$ is the unit step function

Where Δ is defined as (explanation to follow; when a ramp response is considered)

$$\Delta = \frac{1}{K_v} - \frac{M}{\omega_{3dBf}} \qquad \text{EQ 23}$$

Therefore the 10 to 90 % rise time is approximately;

$$t_{r10\_90} = \frac{2.1972}{\frac{\omega_{3dBf}}{M}} \qquad \text{EQ 24}$$

There is no overshoot!

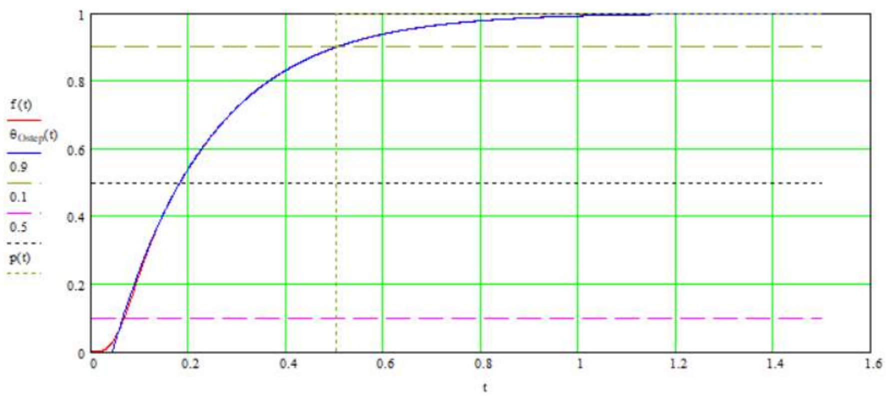
For lower position bandwidths -- in the limit as K_V -> 0, the POS_BW(χ) -> K_V

Example 1 (ω_D = 100 rad/s, χ = 2, K_v = 4.114 s^{-1} for **ω_{3dBf} / 10** – See Table 1).

σ_1 = 5.029 rad/s, σ_2 = 44.971 rad/s, and σ_3 +/- j ω_3 = 25.00 +/- j 40.606 rad/s.

For f(t): a1 = -1.25026, a2 = 0.139818, b1 = 0.110446 and b2 = 0.067999

Note that f(t) is the exact step function response (all poles considered – red trace) and p(t) is just a marker

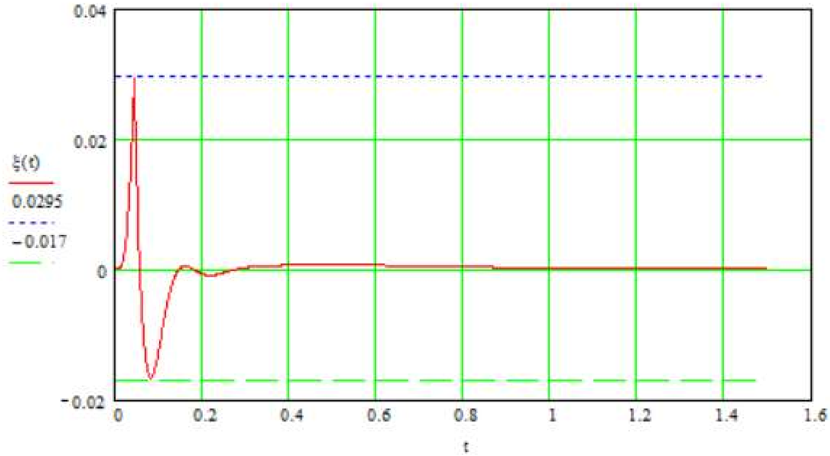


$t_{r10\_90} := 0.5023 \cdot s - 0.06685 \cdot s$

$t_{r10\_90} = 0.43545\,s$

EQ 23 calculates a rise time of estimate of 0.43944 s, ~ 0.92 % greater than the actual.

Error function with $\Delta = (1 / K_v - M / \omega_{3dBf}) =$ 0.043072 s.

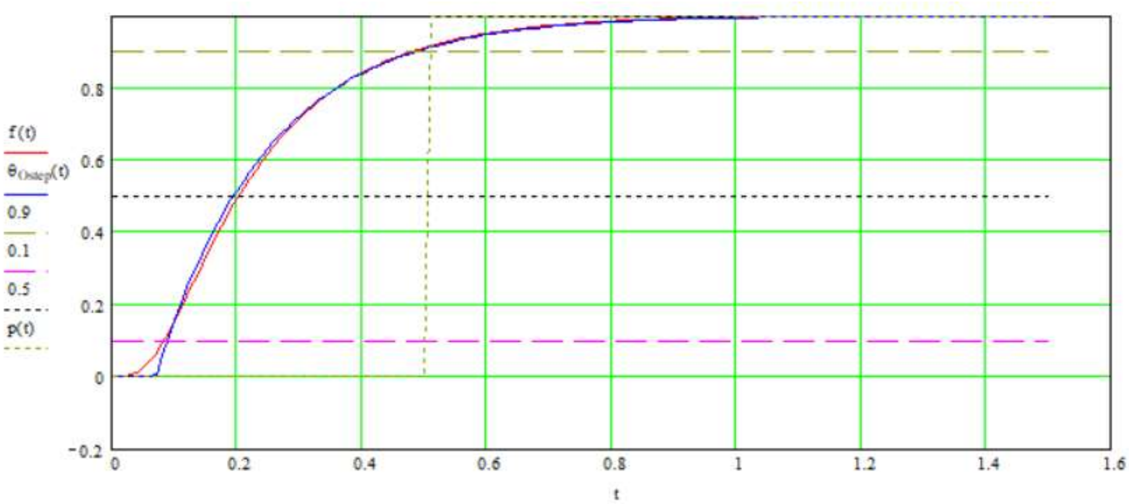


Note that open loop crossover frequency occurs at Kv. Using EQ 12 and EQ 13 one finds that the phase margin is 80.561 deg and the gain margin is 19.17 dB (@ 35.3 rad/s)

Example 2 (ω_D = 100 rad/s, χ = 2.5, K_v = 4.006 s^{-1} for $\boldsymbol{\omega_{3dBf}}$ **/ 5** – See Table 2);

σ_1 = 5.852 rad/s, σ_2 = 26.364 rad/s, and σ_3 +/- j ω_3 = 33.892 +/- j 22.657 rad/s.

For f(t): a1 = -1.643739, a2 = 0.831853, b1 = -0.188114 and b2 = 0.262005



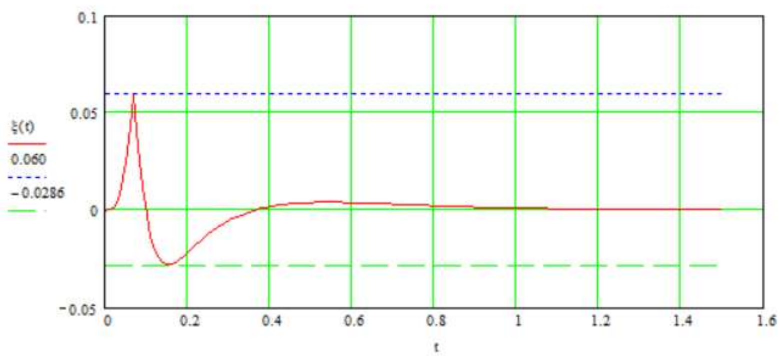
$t_{r10\_90} := 0.479 \cdot s - 0.0829 \cdot s$

$t_{r10\_90} = 0.3961\,s$

EQ 22 calculates a rise time of estimate of 0.3969 s, ~ 0.2 % greater than the actual.

Error function with Δ = (1 / K_v – M / ω_{3dBf}) = 0.069 s.

$$\xi(t) := f(t) - \theta_{Ostep}(t)$$



Open loop crossover frequency occurs at Kv. Using EQ 12 and EQ 13 one finds that the phase margin is 75.651 deg and the gain margin is 18.45 dB (@ 25.25 rad/s)

For the **ramp output response** when using the approximation model, all one needs to do is integrate EQ 22 over the limits of Δ to t which yields:

Assuming zero for the initial conditions.

With θ_x the slope of the ramp input in rad /s

$$\theta_{Oramp}(t) := \theta_x \cdot \left[(t - \Delta) + \frac{M}{\omega_{3dBf}} \left[-1 + \exp\left[\frac{-\omega_{3dBf}}{M} \cdot (t - \Delta) \right] \right] \right] \cdot \Phi(t - \Delta) \qquad \text{EQ 24}$$

Note that the ramp error value is obtained by evaluating:

$$\lim_{t \to \infty} \left[\theta_x \cdot (t) - \theta_{Oramp}(t) \right] = \frac{\theta_x}{K_v}$$

Using the ramp response approximation of EQ 24 and this limit yields:

Repeated -

$$\Delta = \frac{1}{K_v} - \frac{M}{\omega_{3dBf}} \qquad \text{EQ 23}$$

Which was used in EQ 22 above.

Table 1 below summarizes Position Bandwidth errors for EQ 19 using randomly selected data.

Desire a Position BW = 1/10 ω_{3dBf}, for ω_D = 100 rad/s

Limited Span Symmetry Method

No.	χ	*ω3dBf (rad/s)	Kv (rad/s)	POS BW (rad/s)	Percent Error (%)
1	2.000	50.000	4.114	5.0048	0.095
2	2.100	44.370	3.670	4.4423	0.119
3	2.200	39.330	3.271	3.9350	0.051
4	2.250	37.020	3.087	3.7013	-0.018
5	2.300	34.855	2.914	3.4820	-0.088
6	2.375	31.915	2.679	3.1863	-0.165
7	2.400	30.998	2.606	3.0939	-0.191
9	2.435	29.775	2.507	2.9709	-0.221
8	2.500	27.683	2.339	2.7614	-0.250
10	2.716	22.060	1.885	2.2023	-0.168
11	2.850	19.397	1.669	1.9402	0.023
12	2.950	17.719	1.533	1.7759	0.223

Position Bandwidth Error Summary
Table 1

***Actual Velocity Loop Prefiltered Bandwidth Values**

Table 2 below summarizes Position Bandwidth errors for EQ 20 using randomly selected data.

Desire a Position BW = 1/5 ω_{3dBf}, for ω_D = 100 rad/s

Limited Span Symmetry Method

No.	χ	*ω3dBf (rad/s)	Kv (rad/s)	POS BW (rad/s)	Percent Error (%)
1	2.000	50.000	6.851	10.0441	0.441
2	2.100	44.370	6.148	8.9054	0.344
3	2.200	39.330	5.510	7.8729	0.088
4	2.250	37.020	5.215	7.3983	-0.077
5	2.300	34.855	4.937	6.9540	-0.243
6	2.375	31.915	4.557	6.3565	-0.415
7	2.400	30.998	4.438	6.1705	-0.469
9	2.435	29.775	4.279	5.9236	-0.527
8	2.500	27.683	4.006	5.5049	-0.573
10	2.716	22.060	3.266	4.3979	-0.319
11	2.850	19.397	2.912	3.8848	0.140
12	2.950	17.719	2.687	3.5651	0.600

Position Bandwidth Error Summary
Table 2

***Actual Velocity Loop Prefiltered Bandwidth Values**

Phase Step and Ramp Node Response Summary (Approximation Model)

Refer to Figures 1A and 1B

Step of Phase (zero initial conditions)

θ_s - Represents the Input Phase Step Size in rad

Step Input

$$\theta_{Istep}(t) := \theta_s \cdot \Phi(t) \qquad (rad)$$

Position summing point error

$$e_{step}(t) := \theta_s \cdot \Phi(t) - \theta_s \cdot \left[1 - \exp\left[-\left(\frac{\omega_{3dBf}}{M}\right) \cdot (t - \Delta)\right]\right] \cdot \Phi(t - \Delta) \qquad (rad)$$

Velocity Loop Input

$$\omega_{Istep}(t) := \left[\theta_s \cdot \Phi(t) - \theta_s \cdot \left[1 - \exp\left[-\left(\frac{\omega_{3dBf}}{M}\right) \cdot (t - \Delta)\right]\right] \cdot \Phi(t - \Delta)\right] \cdot K_v \qquad \left(\frac{rad}{s}\right)$$

Velocity Loop Output

$$\omega_{Ostep}(t) := \theta_s \cdot \frac{\omega_{3dBf}}{M} \cdot \exp\left[\frac{-\omega_{3dBf}}{M} \cdot (t - \Delta)\right] \cdot \Phi(t - \Delta) \qquad \left(\frac{rad}{s}\right)$$

Postion Loop Output

$$\theta_{Ostep}(t) := \theta_s \cdot \left[1 - \exp\left[-\left(\frac{\omega_{3dBf}}{M}\right) \cdot (t - \Delta)\right]\right] \cdot \Phi(t - \Delta) \qquad (rad)$$

Ramp of Phase (zero initial conditions)

θ_x - Represents the Input Phase Ramp Slope in rad / s.

Ramp Input

$$\theta_{Iramp}(t) := \theta_x \cdot t \cdot \Phi(t) \qquad (rad)$$

Position summing point error

$$e_{ramp}(t) \sim \frac{\theta_x}{K_v}\left[1 - \exp\left[-\left(\frac{\omega_{3dBf}}{M}\right)\cdot t\right]\right]\cdot\Phi(t) \qquad \text{as } \Delta < \frac{1}{K_v}, \frac{M}{\omega_{3dBf}} \qquad (rad)$$

Velocity Loop Input

$$\omega_{Iramp}(t) \sim \theta_x\left[1 - \exp\left[-\left(\frac{\omega_{3dBf}}{M}\right)\cdot t\right]\right]\cdot\Phi(t) \qquad \left(\frac{rad}{s}\right)$$

Velocity Loop Output

$$\omega_{Oramp}(t) := \theta_x\cdot\left[1 - \exp\left[-\left(\frac{\omega_{3dBf}}{M}\right)\cdot(t-\Delta)\right]\right]\cdot\Phi(t-\Delta) \qquad \left(\frac{rad}{s}\right)$$

Position Loop Output

$$\theta_{Oramp}(t) := \theta_x\cdot\left[(t-\Delta) + \frac{M}{\omega_{3dBf}}\cdot\left[-1 + \exp\left[\left(\frac{-\omega_{3dBf}}{M}\right)\cdot(t-\Delta)\right]\right]\right]\cdot\Phi(t-\Delta) \qquad (rad)$$

Some Final Notes:

All parameters are proportional to ω_D, the highest frequency dominant velocity/current open loop pole. Therefore, to scale the parameters used above (such as pole frequencies, ω_{3dBf}, K_v and POS_BW) divide the Table (1 or 2) values by 100 rad/s and multiple by the new ω_D.

Refer to EQ 28 through 31 in the document <u>*Cascade PI Controller Design using a Limited Span Method,*</u> by this author to derive EQ 25 below.

For $M = 5$ or 10

$$POS\_BW(\chi, M) = \frac{\omega_{3dBf}(\chi)}{M} = \frac{u(\chi)\cdot\omega_D}{M} \qquad \text{EQ 25}$$

Another way to think of the tables is to view the ω_{3dBf}, K_v and POS_BW data as percentages and not absolute frequencies. The poles, σ1, σ2, (σ3 – j ω3) and (σ3 + j ω3) are likewise scaled in proportion to ω_D.

$K_v(\chi, M)$ (EQ 19 and 20) are only derived for M = 5 and M = 10. For other values of M simple interpolation using these equations will not work. It is best to use an iterative approach to arrive at the K_v required for a different M.

The important thing is that by using a velocity loop based on a limited span range, a higher position closed loop frequency is achievable.