A Comparison of Smoothing Estimation filters for Servo Motor Applications

This document is intended to compare a set of widely used filters to determine the quality of each regarding noise reduction and application limitations.

The class of filters to be examined are typically used in servo applications for filtering position, velocity, and acceleration data. This is of course not the only application as data tracking applications are employed in various enterprises as well.

The filters used in this analysis are:

1. αβϒ
2. Savitzky-Golay (SG) *[window size of 5 and polynomial degree of 2 or 3]*
3. Numerical Methods
    a. Moving Average (MA)
    b. Numerical Differentiation Approximation
    c. Richardson's Extrapolation

Note that item 3 covers the same calculations made in items 1 and 2.

Symbols:

Symbol	Description	Units
IIR	Infinite Impulse Response Filter	
FIR	Finite Impulse Response Filter	
μ	Mean	
σ	Standard Deviation	
τ	Time Delay	seconds (s)
t	Time	seconds (s)
T_s	Sampling Time	seconds (s)
$\mathbf{x}_i$	Input Variable Function (i)	
$\mathbf{y}$	Output Position Variable	(rad or meters)
$\mathbf{v}$	Output Velocity Variable	(rad /s or meters / s)
$\mathbf{a}$	Output Acceleration Variable	(rad /s /s or meters / s /s)
ω	Angular velocity	(rad/s)
$\phi_j(\omega)$	Phase Angle [function j]	(rad or degrees)
$G_j dB\ (\omega)$	Magnitude [function j]	Decibels (dB)
ω_{3dB}	Loop Bandwidth	(rad/s)

There are eight (8) parts to this document:

Part 1 – Filter Discrete Equations

Part 2 – Filter Gain and Phase Plots

Part 3 – Base Test Functions and Delay Offset.

Part 4 – Aperiodic Noise Calculations.

Part 5 – Periodic Noise Calculations – αβϒ with Sine Input

Part 6 – Bandwidth and Noise Dependance on αβϒ Parameters

Part 7 – Observations and Conclusions

Part 8 -- References

Part 1 – Filter Equations

αβγ Filter - Position Section

$$y_{k+3} = \alpha \cdot x_i(k+3) + \left(-2\cdot\alpha + \beta + \frac{\gamma}{4}\right)\cdot x_i(k+2) + \left(\alpha - \beta + \frac{\gamma}{4}\right)\cdot x_i(k+1)$$
$$-\left(\alpha + \beta + \frac{\gamma}{4} - 3\right)\cdot y_{k+2} - \left(-2\cdot\alpha - \beta + \frac{\gamma}{4} + 3\right)\cdot y_{k+1} - (\alpha - 1)\cdot y_k \qquad \textbf{EQ 1}$$

αβγ Filter - Velocity Section

$$b_{k+3} = \beta \cdot x_i(k+2) + \left(-2\cdot\beta + \frac{\gamma}{2}\right)\cdot x_i(k+1) + \left(\beta - \frac{\gamma}{2}\right)\cdot x_i(k)$$
$$-\left(\alpha + \beta + \frac{\gamma}{4} - 3\right)\cdot b_{k+2} - \left(-2\cdot\alpha - \beta + \frac{\gamma}{4} + 3\right)\cdot b_{k+1} - (\alpha - 1)\cdot b_k \qquad \textbf{EQ 2a}$$

$$v_{k+3} = b_{k+3}\cdot\frac{1\cdot s}{T_s} \qquad \textbf{EQ 2b}$$

αβγ Filter - Acceleration Section

$$c_{k+3} = \frac{\gamma}{2}\cdot x_i(k+3) - \gamma\cdot x_i(k+2) + \frac{\gamma}{2}\cdot x_i(k+1)$$
$$-\left(\alpha + \beta + \frac{\gamma}{4} - 3\right)\cdot c_{k+2} - \left(-2\cdot\alpha - \beta + \frac{\gamma}{4} + 3\right)\cdot c_{k+1} - (\alpha - 1)\cdot c_k \qquad \textbf{EQ 3a}$$

$$a_{k+3} = c_{k+3}\cdot\left(\frac{1\cdot s}{T_s}\right)^2 \qquad \textbf{EQ 3b}$$

Savitzky-Golay Filter (SG) - Position Section

$$y_{SG_{k+2}} = \frac{1}{35}\cdot\left(-3\cdot x_i(k) + 12\cdot x_i(k+1) + 17\cdot x_i(k+2) + 12\cdot x_i(k+3) - 3\cdot x_i(k+4)\right) \qquad \textbf{EQ 4}$$

Savitzky-Golay Filter (SG) - Velocity Section

$$v_{SG_{k+2}} = \frac{1}{10}\cdot\left(-2\cdot x_i(k) - 1\cdot x_i(k+1) + 0\cdot x_i(k+2) + 1\cdot x_1(k+3) + 2\cdot x_1(k+4)\right)\cdot\frac{1\cdot s}{T_s} \qquad \textbf{EQ 5}$$

Savitzky-Golay (SG) Filter - Acceleration Section

$$a_{SG_{k+2}} = \frac{1}{7}\cdot\left[\left(2\cdot x_i(k) - 1\cdot x_i(k+1) + -2\cdot x_i(k+2) + -1\cdot x_i(k+3) + 2\cdot x_i(k+4)\right)\cdot\frac{1\cdot s^2}{T_s^{\,2}}\right] \qquad \textbf{EQ 6}$$

Numeric - Position *3 pt Simple Moving Average*

$$y\_num_{k+1} = \frac{x_i(k+2) + x_i(k+1) + x_i(k)}{3} \qquad \textbf{EQ 7}$$

Numeric First Derivative Approximation Filter - Velocity

$$v\_num_{k+1} = \frac{x_i(k+2) - x_i(k)}{2\cdot T_s} \qquad \textbf{EQ 8}$$

Numeric First Derivative Richardson's Extrapolation - Velocity
Same as a 1st derivative SG of window size 5 and polynomial degree of 4/5

$$v\_Rch_{k+2} = \frac{1}{12}\cdot\left[\left(-x_1(k+4)\right) + 8\cdot x_1(k+3) - 8\cdot x_1(k+1) + x_1(k)\right]\cdot\frac{1\cdot s}{T_s} \qquad \textbf{EQ 9}$$

Numeric Second Derivative Approximation Filter - Acceleration

$$a\_num_{k+1} = \frac{x_i(k+2) - 2\cdot x_i(k+1) + x_i(k)}{T_s^{\,2}}\cdot 1\cdot s^2 \qquad \textbf{EQ 10}$$

Numeric Second Derivative Richardson's Extrapolation - Acceleration
Same as a 2nd derivative SG of window size 5 and polynomial degree of 4/5

$$a\_Rch_{k+2} = \frac{1}{12}\cdot\left(-x_3(k) + 16\cdot x_3(k+1) - 30\cdot x_3(k+2) + 16\cdot x_3(k+3) - 1\cdot x_3(k+4)\right)\cdot\frac{s^2}{T_s^{\,2}} \qquad \textbf{EQ 11}$$

Part 2 - Filter Gain and Phase Plots

$\alpha := 0.300 \qquad \beta := 0.0356 \qquad \gamma := 0.00141$

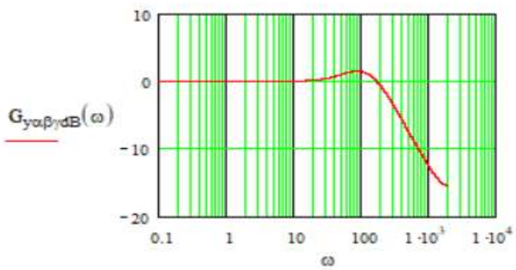
and Ts = 0.0016 s

αβγ Filter - Position Section

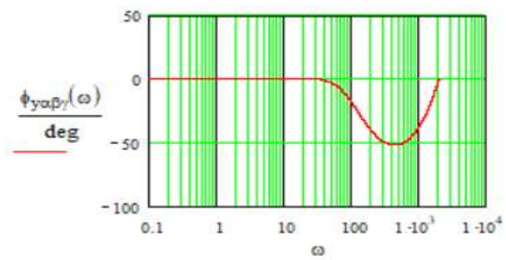
$$G_{y\alpha\beta\gamma}(\omega) := \frac{\exp(j\cdot\omega\cdot T_s)\cdot\left[\alpha\cdot(\exp(j\cdot\omega\cdot T_s)-1)^2+\left(\beta+\frac{\gamma}{4}\right)\cdot(\exp(j\cdot\omega\cdot T_s)-1)+\frac{\gamma}{2}\right]}{\exp(j\cdot\omega\cdot T_s)^3+\left(\alpha+\beta+\frac{\gamma}{4}-3\right)\cdot\exp(j\cdot\omega\cdot T_s)^2+\left(-2\cdot\alpha-\beta+\frac{\gamma}{4}+3\right)\cdot\exp(j\cdot\omega\cdot T_s)+(\alpha-1)} \qquad \textbf{EQ 12}$$

$G_{y\alpha\beta\gamma dB}(\omega) := 20\cdot\log(|G_{y\alpha\beta\gamma}(\omega)|)$ Gain Response

$\phi_{y\alpha\beta\gamma}(\omega) := \arg(G_{y\alpha\beta\gamma}(\omega))$ Phase Response



GAIN Plot
Figure 1 a



PHASE Plot
Figure 1 b

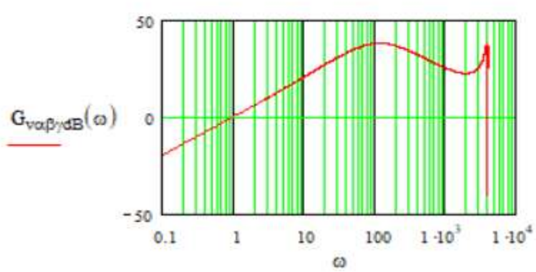
αβγ Filter - Velocity Section

$Z(\omega) := \exp(j\cdot\omega\cdot T_s)$

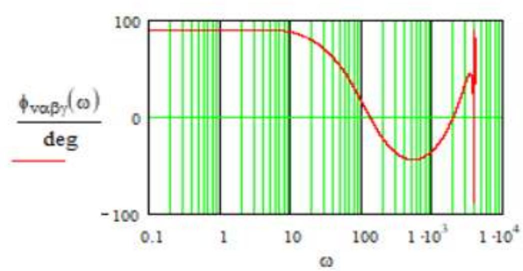
$$G_{v\alpha\beta\gamma}(\omega) := \frac{Z(\omega)\cdot\left[\beta\cdot(Z(\omega)-1)^2+\frac{\gamma}{2}\cdot(Z(\omega)-1)\right]}{\left[Z(\omega)^3+\left(\alpha+\beta+\frac{\gamma}{4}-3\right)\cdot Z(\omega)^2+\left(-2\cdot\alpha-\beta+\frac{\gamma}{4}+3\right)\cdot Z(\omega)+(\alpha-1)\right]}\cdot\frac{1\cdot s}{T_s} \qquad \textbf{EQ 13}$$

$G_{v\alpha\beta\gamma dB}(\omega) := 20\cdot\log(|G_{v\alpha\beta\gamma}(\omega)|)$ Gain Response

$\phi_{v\alpha\beta\gamma}(\omega) := \arg(G_{v\alpha\beta\gamma}(\omega))$ Phase Response



GAIN Plot
Figure 2 a



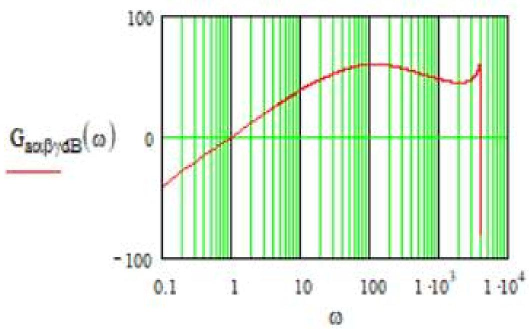
PHASE Plot
Figure 2 b

αβγ Filter - Acceleration Section

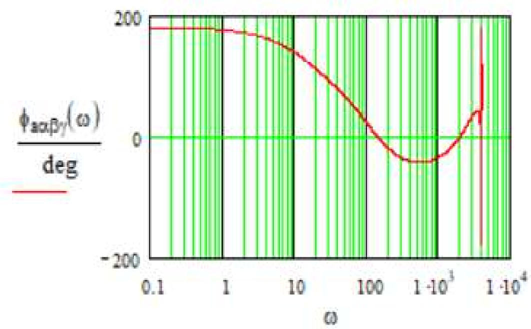
$$G_{a\alpha\beta\gamma}(\omega) := \frac{Z(\omega)\cdot\left[\frac{\gamma}{2}\cdot(Z(\omega)-1)^2\right]}{Z(\omega)^3+\left(\alpha+\beta+\frac{\gamma}{4}-3\right)\cdot Z(\omega)^2+\left(-2\cdot\alpha-\beta+\frac{\gamma}{4}+3\right)\cdot Z(\omega)+(\alpha-1)}\cdot\left(\frac{1\cdot s}{T_s}\right)^2 \qquad \textbf{EQ 14}$$

$G_{a\alpha\beta\gamma dB}(\omega) := 20\cdot\log(|G_{a\alpha\beta\gamma}(\omega)|)$ Gain Response

$\phi_{a\alpha\beta\gamma}(\omega) := \arg(G_{a\alpha\beta\gamma}(\omega))$ Phase Response



GAIN Plot
Figure 3 a



PHASE Plot
Figure 3 b

The 3 dB Bandwith for each αβγ filter section with

$\alpha = 0.3 \qquad \beta = 0.0356 \qquad \gamma = 1.41\times10^{-3}$

is given by:

$$\omega_{y\_\alpha\beta\gamma\_3dB} = \frac{0.44}{T_s} \qquad \omega_{v\_\alpha\beta\gamma\_3dB} = \frac{0.1784}{T_s} \qquad \omega_{a\_\alpha\beta\gamma\_3dB} = \frac{0.02396}{T_s}$$

Only plots with Ts = 0.0016 s are shown, but the 3 dB relationships are valid for any Ts, with the same αβϒ parameters used herein

$$G_{y\alpha\beta\gamma dB}(\omega_{y\_\alpha\beta\gamma\_3dB}) = -3.0009\,dB$$

$$G_{v\alpha\beta\gamma dB}(\omega_{v\_\alpha\beta\gamma\_3dB}) - 20\cdot\log\left(\left|j\cdot\frac{\omega_{v\_\alpha\beta\gamma\_3dB}}{1\cdot\frac{rad}{s}}\right|\right) = -3.0019\,dB$$

$$G_{a\alpha\beta\gamma dB}(\omega_{a\_\alpha\beta\gamma\_3dB}) - 40\cdot\log\left(\left|j\cdot\frac{\omega_{a\_\alpha\beta\gamma\_3dB}}{1\cdot\frac{rad}{s}}\right|\right) = -2.9998\,dB$$

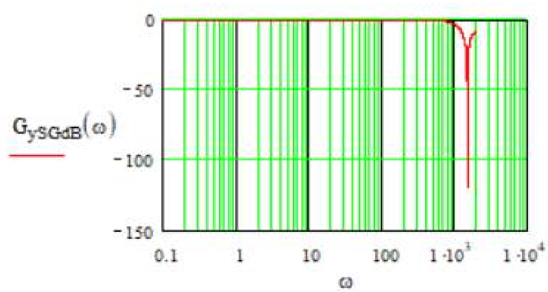
SG Filter - Position Section

$$Z(\omega) := \exp(j \cdot \omega \cdot T_s)$$

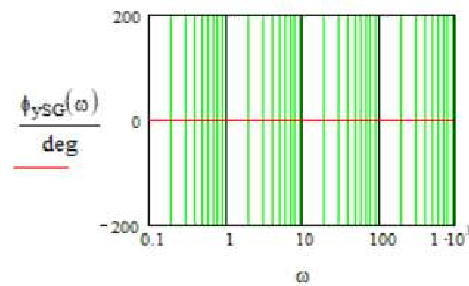
$$G_{ySG}(\omega) := \frac{1}{35} \cdot \frac{\left(-3 \cdot Z(\omega)^4 + 12 \cdot Z(\omega)^3 + 17 \cdot Z(\omega)^2 + 12 \cdot Z(\omega)^1 - 3\right)}{Z(\omega)^2} \qquad \textbf{EQ 15}$$

$$G_{ySGdB}(\omega) := 20 \cdot \log\left(\left|G_{ySG}(\omega)\right|\right) \qquad \text{Gain Response}$$

$$\phi_{ySG}(\omega) := \arg\left(G_{ySG}(\omega)\right) \qquad \text{Phase Response}$$



GAIN Plot
Figure 4 a



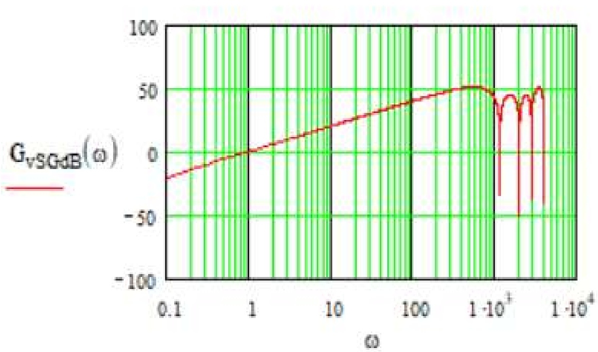
PHASE Plot
Figure 4 b

SG Filter - Velocity Section

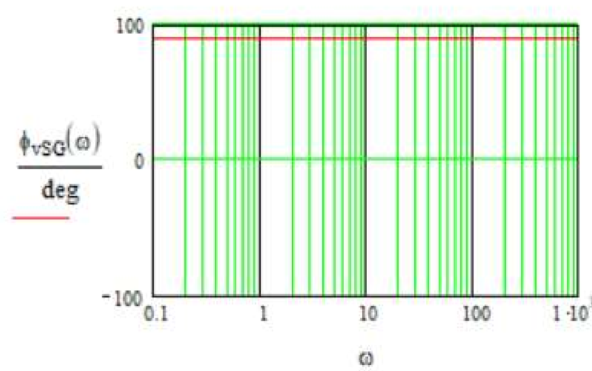
$$G_{vSG}(\omega) := \frac{1}{10} \cdot \frac{\left(2 \cdot Z(\omega)^4 + Z(\omega)^3 - Z(\omega) - 2\right)}{Z(\omega)^2} \cdot \frac{1 \cdot s}{T_s} \qquad \textbf{EQ 16}$$

$$G_{vSGdB}(\omega) := 20 \cdot \log\left(\left|G_{vSG}(\omega)\right|\right) \qquad \text{Gain Response}$$

$$\phi_{vSG}(\omega) := \arg\left(G_{vSG}(\omega)\right) \qquad \text{Phase Response}$$



GAIN Plot
Figure 5 a



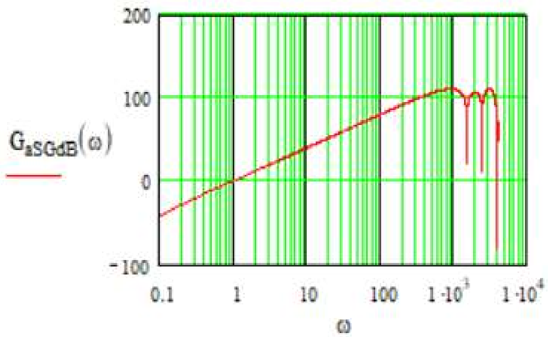
PHASE Plot
Figure 5 b

SG Filter - Acceleration Section

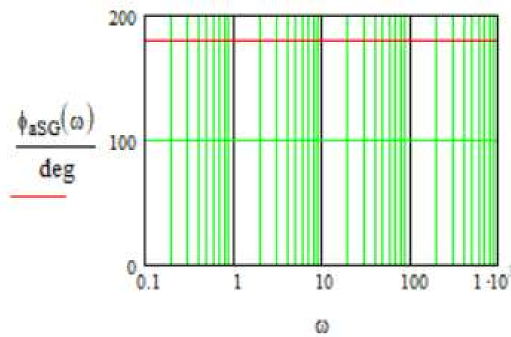
$$G_{aSG}(\omega) := \frac{1}{7} \cdot \frac{\left(2 \cdot Z(\omega)^4 - Z(\omega)^3 - 2 \cdot Z(\omega)^2\right) - 1 \cdot Z(\omega)^1 + 2}{Z(\omega)^2} \cdot \left(\frac{1 \cdot s}{T_s}\right)^2 \qquad \textbf{EQ 17}$$

$$G_{aSGdB}(\omega) := 20 \cdot \log\left(\left|G_{aSG}(\omega)\right|\right)$$ Gain Response

$$\phi_{aSG}(\omega) := \left|\arg\left(G_{aSG}(\omega)\right)\right|$$ Phase Response



GAIN Plot
Figure 6 a



PHASE Plot
Figure 6 b

The 3 dB Bandwith for each SG filter section with is given by:

$$\omega_{y\_SG\_3dB} = \frac{1.4937}{T_s} \qquad \omega_{v\_SG\_3dB} = \frac{0.7585}{T_s} \qquad \omega_{a\_SG\_3dB} = \frac{0.9454}{T_s}$$

Note that the acceleration bandwidth is larger than the velocity bandwidth!

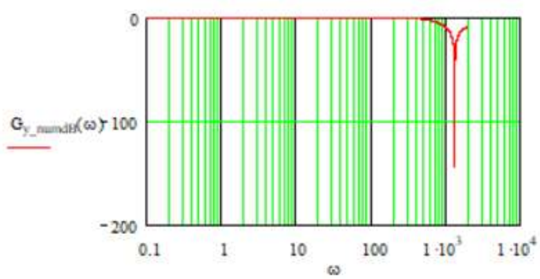
Numeric - Position Section

$$Z(\omega) := \exp(j \cdot \omega \cdot T_s)$$

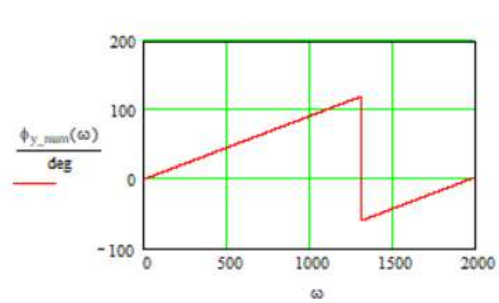
$$G_{y\_num}(\omega) := \frac{Z(\omega)^2 + Z(\omega)^1 + 1}{3} \qquad \textbf{EQ 18}$$

$$G_{y\_numdB}(\omega) := 20 \cdot \log\left(\left|G_{y\_num}(\omega)\right|\right)$$ Gain Response

$$\phi_{y\_num}(\omega) := \arg\left(G_{y\_num}(\omega)\right)$$ Phase Response



GAIN Plot
Figure 7 a



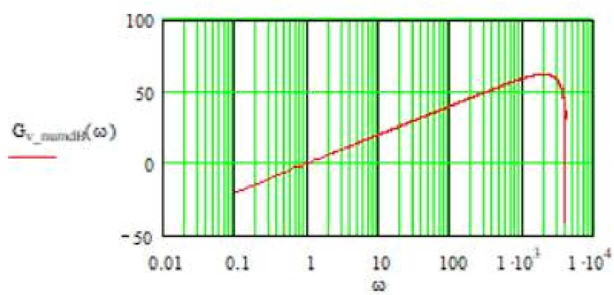
PHASE Plot
Figure 7 b

Numeric First Derivative Section - Velocity

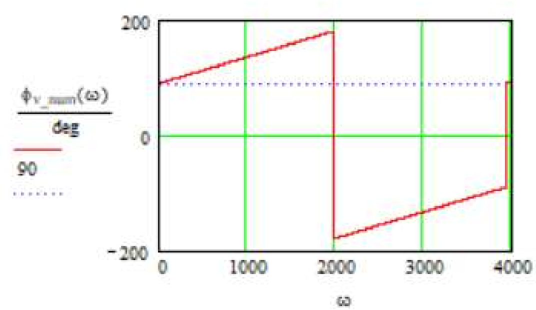
$$G_{v\_num}(\omega) := \frac{\left(Z(\omega)^2 - 1\right)}{2}\cdot\frac{1\cdot s}{T_s}$$ **EQ 19**

$$G_{v\_numdB}(\omega) := 20\cdot\log\left(\left|G_{vv\_num}(\omega)\right|\right)$$ Gain Response

$$\phi_{v\_num}(\omega) := \arg\left(G_{v\_num}(\omega)\right)$$ Phase Response



GAIN Plot
Figure 8 a



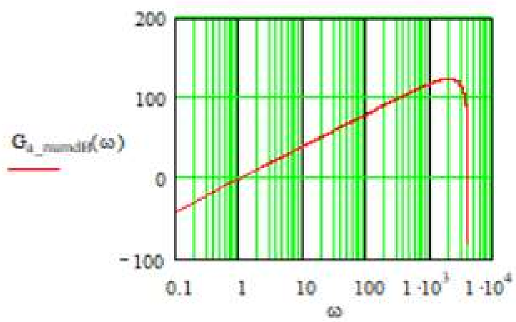
PHASE Plot
Figure 8 b

Numeric Second Derivative Section - Acceleration

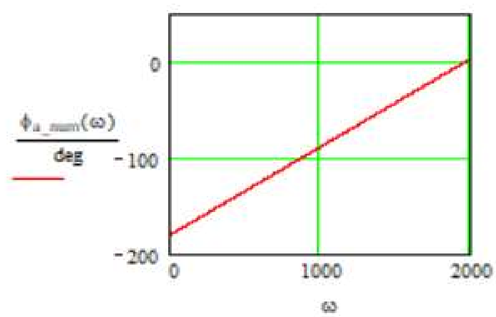
$$G_{a\_num}(\omega) := \left(Z(\omega)^2 - 2\cdot Z(\omega) + 1\right)\cdot\left(\frac{1\cdot s}{T_s}\right)^2$$ **EQ 20**

$$G_{a\_numdB}(\omega) := 20\cdot\log\left(\left|G_{a\_num}(\omega)\right|\right)$$ Gain Response

$$\phi_{a\_num}(\omega) := \arg\left(G_{a\_num}(\omega)\right)$$ Phase Response



GAIN Plot
Figure 9 a



PHASE Plot
Figure 9 b

The 3 dB Bandwith for each Numeric filter section

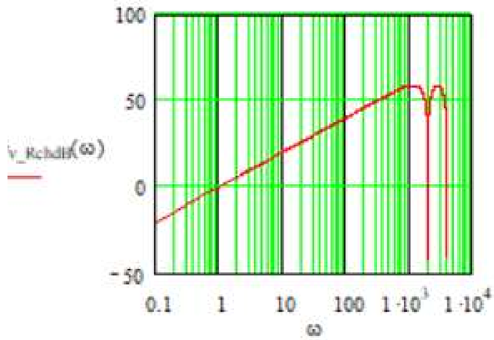
$$\omega_{y\_num\_3dB} = \frac{0.9741}{T_s} \qquad \omega_{v\_num\_3dB} = \frac{1.3893}{T_s} \qquad \omega_{a\_num\_3dB} = \frac{2.0005}{T_s}$$

Numeric Richardson's First Derivative - Velocity Section

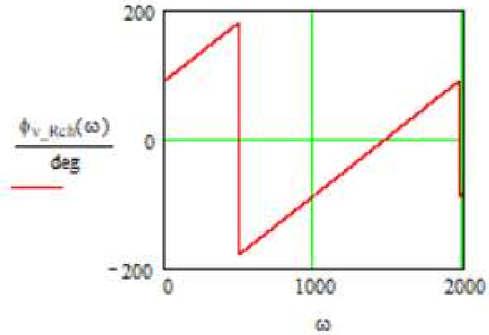
$$G_{v\_Rch}(\omega) := \frac{1}{12}\cdot\left(-Z(\omega)^4 + 8\cdot Z(\omega)^3 - 8\cdot Z(\omega) + 1\right)\cdot\frac{s}{T_s}$$ **EQ 21**

$$G_{v\_RchdB}(\omega) := 20\cdot\log\left(\left|G_{v\_Rch}(\omega)\right|\right)$$ Gain Response

$$\phi_{v\_Rch}(\omega) := \arg\left(G_{v\_Rch}(\omega)\right)$$ Phase Response



GAIN Plot
Figure 10 a



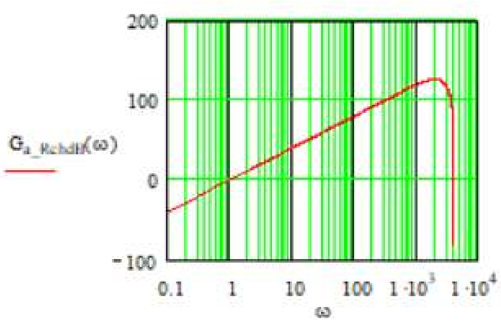
PHASE Plot
Figure 10 b

Numeric Richardson's Second Derivative - Acceleration Section

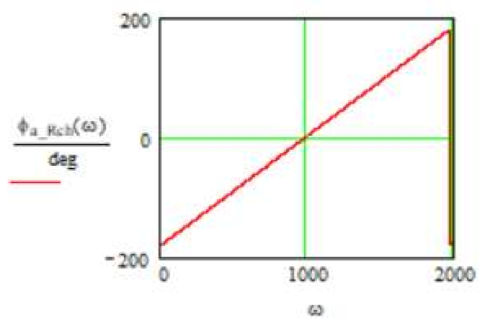
$$G_{a\_Rch}(\omega) := \left[\frac{1}{12}\cdot\left(-1 + 16\cdot Z(\omega) - 30\cdot Z(\omega)^2 + 16\cdot Z(\omega)^3 - Z(\omega)^4\right)\right]\cdot\left(\frac{1\cdot s}{T_s}\right)^2$$ **EQ 22**

$$G_{a\_RchdB}(\omega) := 20\cdot\log\left(\left|G_{a\_Rch}(\omega)\right|\right)$$ Gain Response

$$\phi_{a\_Rch}(\omega) := \arg\left(G_{a\_Rch}(\omega)\right)$$ Phase Response



GAIN Plot
Figure 11 a



PHASE Plot
Figure 11 b

The 3 dB Bandwith for each Richardson's section

$$\omega_{v\_Rch\_3dB} = \frac{1.9208}{T_s} \qquad \omega_{a\_Rch\_3dB} = \frac{2.637}{T_s}$$

Part 3 – Base Test Functions and Delay Offset

Amplitude Values Used

$$Y := 10 \cdot \text{units}$$

$$V := 10 \cdot \frac{\text{units}}{s}$$

$$A := 10 \cdot \frac{\text{units}}{s^2}$$

$$J := 100 \cdot \frac{\text{units}}{s^3}$$

$$A_p := 5 \cdot \text{units}$$

		Signal type
$STEP(k) := Y \cdot \Phi(k)$	10 units	CONSTANT
$RAMP(k) := V \cdot k \cdot T_s \cdot \Phi(k)$	$10 \cdot \frac{\text{units}}{s}$	LINEAR
$PARB(k) := \frac{A}{2} \cdot (k \cdot T_s)^2 \cdot \Phi(k)$	$10 \cdot \frac{\text{units}}{s^2}$	QUADRATIC
$CUBC(k) := \frac{J}{6} \cdot (k \cdot T_s)^3 \cdot \Phi(k)$	$100 \cdot \frac{\text{units}}{s^3}$	CUBIC
$SIN(k) := A_p \cdot \sin(k \cdot T_s \cdot \omega_{sig}) \cdot \Phi(k)$	$5 \cdot \text{units}$	SINUSOID

Units are typically radians (rad) or meters (m)

For an Input F(k)

Filter Section	Output	
Position Filter	$F(k)$	units
Velocity Filter	$\frac{d}{dk}F(k)$	$\frac{\text{units}}{s}$
Acceleration Filter	$\frac{d^2}{dk^2}F(k)$	$\frac{\text{units}}{s^2}$

For Ideal Filter Sections (No noise) the following will be true (initial conditions = 0)

Position Filter Output y_k =

units

$$STEP(k) := Y$$ Step Input

$$RAMP(k) := V\cdot(k\cdot T_s)$$ Ramp Input

$$PARB(k) := \frac{A}{2}\cdot(k\cdot T_s)^2$$ Parabolic Input

$$CUBC(k) := \frac{J}{6}\cdot(k\cdot T_s)^3$$ Cubic Input

$$SIN(k) := A_p\cdot\sin(k\cdot T_s\cdot\omega_{sig})$$ Sinusoidal Input

Velocity Filter Output v_k =

$\frac{\text{units}}{s}$

$$\frac{d}{dk}STEP(k) = 0$$ Step Input

$$\frac{d}{dk}RAMP(k) = V$$ Ramp Input

$$\frac{d}{dk}PARB(k) = A\cdot(k\cdot T_s)$$ Parabolic Input

$$\frac{d}{dk}CUBC(k) = \frac{J}{2}\cdot(k\cdot T_s)^2$$ Cubic Input

$$\frac{d}{dk}SIN(k) = A_p\cdot\omega_{sig}\cdot\cos(k\cdot T_s\cdot\omega_{sig})$$ Sinusoidal Input

Acceleration Filter Output a_k =

$$\frac{d^2}{dk^2}STEP(k) = 0$$ Step Input

$$\frac{d^2}{dk^2}RAMP(k) = 0$$ Ramp Input

$$\frac{d^2}{dk^2}\frac{PARP(k)}{2} = A$$ Parabolic Input $\frac{units}{s^2}$

$$\frac{d^2}{dk^2}\frac{CUBC(k)}{6} = J\cdot(k\cdot T_s)$$ Cubic Input

$$\frac{d^2}{dk^2}SIN(k) = -A_p\cdot\omega_{sig}{}^2\cdot\sin(k\cdot T_s\cdot\omega_{sig})$$ Sinusoidal Input

Test Functions W/WO noise

$on := 1$ for additive noise $on := 0$ for additive noise off.

$x_{0\_init} := 0$ $x_{1\_init} := 0$ $x_{2\_init} := 0$ $x_{3\_init} := 0$ $x_{4\_init} := 0$

$$x_0(k) := (x_{0\_init} + STEP(k-3) + on\cdot Var_k)\cdot\Phi(k-3)$$ Note the 3 sample delay

$$x_1(k) := (x_{1\_init} + RAMP(k-3) + on\cdot Var_k)\cdot\Phi(k-3)$$

$$x_2(k) := (x_{2\_init} + PARB(k-3) + on\cdot Var_k)\cdot\Phi(k-3)$$ dimensions are units

$$x_3(k) := (x_{3\_init} + CUBC(k-3) + on\cdot Var_k)\cdot\Phi(k-3)$$

$$x_4(k) := (x_{4\_init} + SIN(k-3) + on\cdot Var_k)\cdot\Phi(k-3)$$

Let

$funct_y(k)$ represents RAMP(k), PARB(k), CUBC(k) or SIN(k)

$funct_v(k)$ represents PARB(k), CUBC(k) or COS(k)

$funct_a(k)$ represents CUBC(k) or ISIN(k)

Where z represents $\alpha\beta\gamma$ or SG or numeric filter type

When the function functy(),functv() or functa() results in a constant steady state output, the noise is best calulated using the difference between the values of yk, vk or ak and respective final value. The rise and settling times may dominate so you must avoid any transient regions when setting summation limits in noise calculations.

Instead of

$$y_k - \text{funct}_y\left[\left(k - k_{y\_z\_delay}(\omega)\right)\cdot T_s\right] \quad \text{use} \quad y_k - Y$$

$$v_k - \text{funct}_v\left[\left(k - k_{v\_z\_delay}(\omega)\right)\cdot T_s\right] \quad \text{use} \quad v_k - A\cdot s\cdot T_s$$

$$a_k - \text{funct}_a\left[\left(k - k_{a\_z\_delay}(\omega)\right)\cdot T_s\right] \quad \text{use} \quad a_k - J\cdot s^2\cdot T_s$$

Note multiplication by s or s2 (seconds) is use to make quantity dimentionless.

Function delays for position, velocity and acceleration filter sections are defined as

$$k_{y\_z\_delay}(\omega) \qquad k_{V\_z\_delay}(\omega) \qquad k_{V\_z\_delay}(\omega)$$

Now to a discussion of time offset delays.

Whether a non-perodic (or aperiodic) input or a periodic input is used there will be different delays between the input and ouput of the filters.

All of the FIR type filters will have a fixed offset relative to the input function, because for a zero or linear linear phase the delay is independent of frequency. Therefore *all filter sections except the αβϒ* (an IIR filter) will only require a fixed delay of 3. Note that all inputs are delayed by the same ampunt to match the filter section output.

For $z \neq \alpha\beta\gamma$

$$k_{y\_z\_delay} = 3$$

$$k_{v\_z\_delay} = 3$$

$$k_{v\_z\_delay} = 3$$

The αβϒ sections are a little more complicated.

First the phase delay of each αβϒ filter section can be found from:

$$\tau_{y\alpha\beta\gamma}(\omega) := \frac{-\phi_{y\alpha\beta\gamma}(\omega)}{\omega} \qquad \textbf{EQ 23}$$

$$\tau_{v\alpha\beta\gamma}(\omega) := \frac{-\phi_{v\alpha\beta\gamma}(\omega) + \frac{\pi}{2}}{\omega} \qquad \textbf{EQ 24}$$

$$\tau_{a\alpha\beta\gamma}(\omega) := \frac{-\phi_{a\alpha\beta\gamma}(\omega) + \pi}{\omega} \qquad \textbf{EQ 25}$$

The delay time in units of k is determined by a constant term (discrete offset) plus the phase offset contribution as follows:

$$k_{y\_\alpha\beta\gamma\_delay}(\omega) := \left(3 + \frac{\tau_{y\alpha\beta\gamma}(\omega)}{T_s}\right) \qquad \textbf{EQ 26}$$

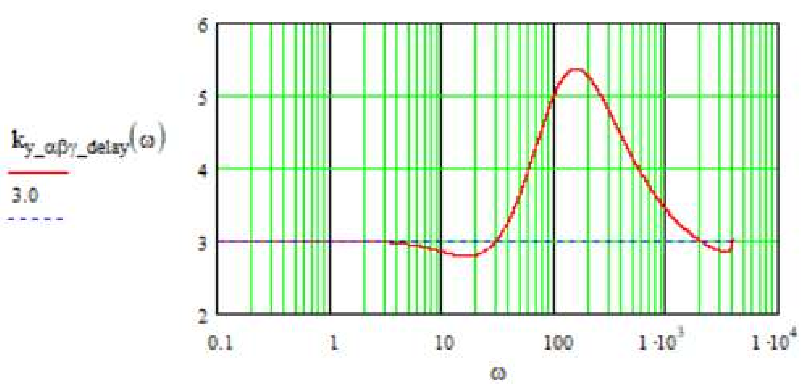
$$k_{v\_\alpha\beta\gamma\_delay}(\omega) := \left(4 + \frac{\tau_{v\alpha\beta\gamma}(\omega)}{T_s}\right) \qquad \textbf{EQ 27}$$

$$k_{a\_\alpha\beta\gamma\_delay}(\omega) := \left(3 + \frac{\tau_{a\alpha\beta\gamma}(\omega)}{T_s}\right) \qquad \textbf{EQ 28}$$

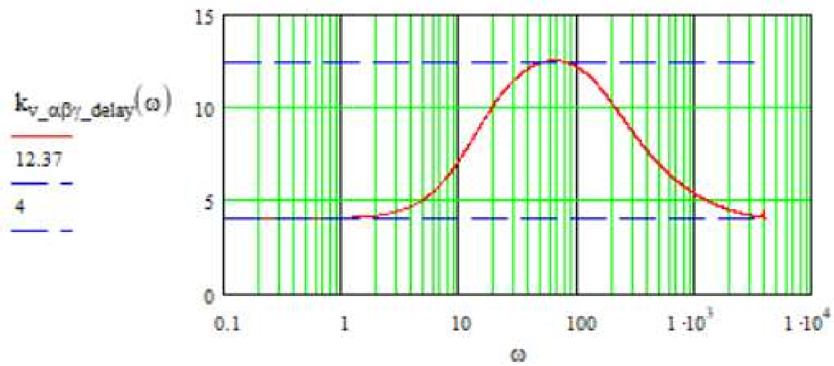
With α = 0.3

For small values of ω (low frequency content)

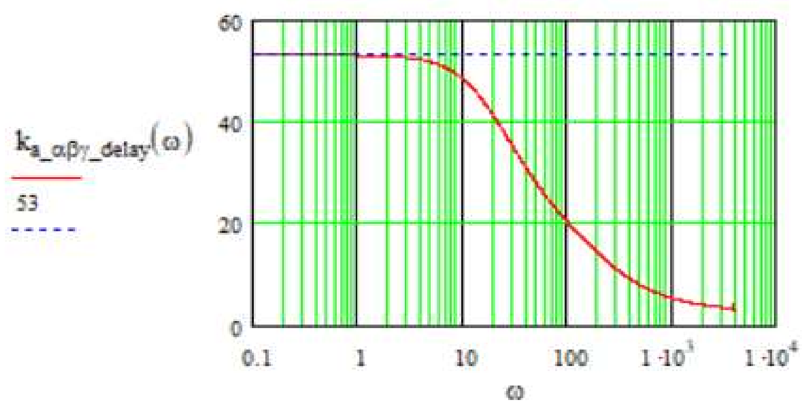
$$k_{y\_\alpha\beta\gamma\_delay}\left(0.5\cdot\frac{rad}{s}\right) = 3 \qquad k_{v\_\alpha\beta\gamma\_delay}\left(0.5\cdot\frac{rad}{s}\right) = 4 \qquad k_{a\_\alpha\beta\gamma\_delay}\left(0.5\cdot\frac{rad}{s}\right) = 53$$



αβϒ Time Position Delay Compensation
Figure 12



αβϒ Time Velocity Delay Compensation
Figure 13



αβϒ Time Velocity Delay Compensation
Figure 14

Note that the above delay curves are a function of α (bandwidth). Shown is α = 0.3.
The equations E 23 through E 28 are valid for 0.3 ≤ α ≤ 0.75.

At any frequency there has to be an adjustment for the amplitue at that frequency. What follows is the way to accomplish this:

Define:

$$\mathrm{Ampl}_y(\omega) := 10^{\frac{G_{ydB}(\omega)}{20}} \qquad \textbf{EQ 29}$$

$$\mathrm{Ampl}_v(\omega) := 10^{\left(\frac{G_{vdB}(\omega)-20\cdot\log\left(\left|j\cdot\frac{\omega}{1\cdot\frac{rad}{s}}\right|\right)}{20}\right)} \qquad \textbf{EQ 30}$$

$$\mathrm{Ampl}_a(\omega) := 10^{\left(\frac{G_{adB}(\omega)-40\cdot\log\left(\left|j\cdot\frac{\omega}{1\cdot\frac{rad}{s}}\right|\right)}{20}\right)} \qquad \textbf{EQ 31}$$

These terms provide the final signal amplitude as a result of multiplying these terms with the ideal output magnitude.

An example is presented in the Periodic Signal Noise Analysis Section.

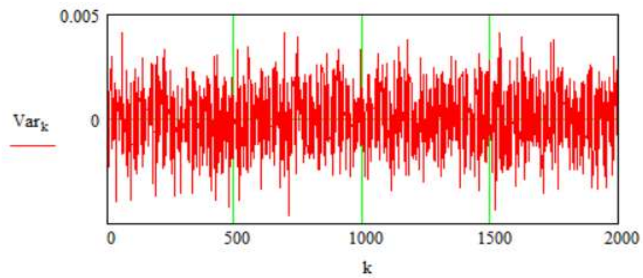
Part 4 - Aperiodic Noise Calculations.

Arbitrary Additive Noise level used throughout

Additive Noise Function (normal distribution zero mean)

$SDevJitter := 4$

$$Var_k := \frac{0.174 \cdot \left(1 \cdot rnorm(1, 0, SDevJitter)_0\right)}{2^9}$$



$$\mu_{var} := \frac{1}{2000} \cdot \sum_{k=1}^{2000} Var_k \qquad \mu_{var} = -1.0511 \times 10^{-5} \quad \text{units}$$

$$\sigma_{var} := \sqrt{\frac{1}{2000}\left[\sum_{k=1}^{2000} \left(Var_k - \mu_{var}\right)^2\right]} \qquad \sigma_{var} = 1.3351 \times 10^{-3} \quad \text{units}$$

Nomenclature

Filter Type = y (Position) , v (Velocity) , a (Acceleration)

Filter Deisignation = $\alpha\beta\gamma$, SG , num for numeric, Rch for Richardson

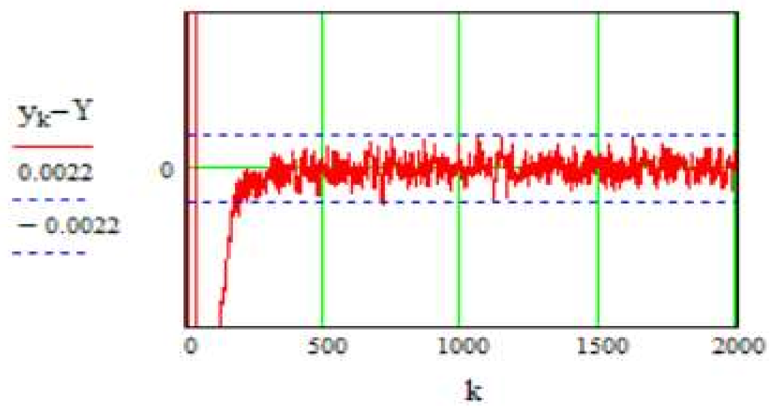
Input = S (STEP) , R (RAMP) , P (Parabola) , C (Cubic) , SIN (Sine)

		Examples
μFilter Type)_(Filter Designation)_(Input)	**mean**	$\mu_{a\_num\_C}$
σFilter Type)_(Filter Designation)_(Input)	**standard deviation**	$\sigma_{y\_SG\_P}$

Aperiodic Input Signal with Noise

Position α-β-γ **Step Input**

$$y_{k+3} := \alpha \cdot x_0(k+3) + \left(-2\cdot\alpha + \beta + \frac{\gamma}{4}\right)\cdot x_0(k+2) + \left(\alpha - \beta + \frac{\gamma}{4}\right)\cdot x_0(k+1) - \left(\alpha + \beta + \frac{\gamma}{4} - 3\right)\cdot y_{k+2} - \left(-2\cdot\alpha - \beta + \frac{\gamma}{4} + 3\right)\cdot y_{k+1} - (\alpha - 1)\cdot y_k$$



αβϒ Position Noise Output
Figure 15

$$\mu_{y\_\alpha\beta\gamma\_S} := \frac{1}{1400}\cdot\sum_{k=501}^{1900}(y_k - Y)$$

$$\mu_{y\_\alpha\beta\gamma\_S} = 3.5668 \times 10^{-5}$$

$$\sigma_{y\_\alpha\beta\gamma\_S} := \sqrt{\frac{1}{1400}\cdot\sum_{k=501}^{1900}\left[(y_k - Y) - \mu_{y\_\alpha\beta\gamma\_S}\right]^2}$$

$$\sigma_{y\_\alpha\beta\gamma\_S} = 6.4062 \times 10^{-4}$$

$$20\cdot\log\left(\frac{\sigma_{y\_\alpha\beta\gamma\_S}}{\sigma_{var}}\right) = -6.38\,\text{dB}$$

Noise Reduction

Position α-β-γ **CUBIC Input**

$$y_{k+3} := \alpha \cdot x_3(k+3) + \left(-2\cdot\alpha + \beta + \frac{\gamma}{4}\right)\cdot x_3(k+2) + \left(\alpha - \beta + \frac{\gamma}{4}\right)\cdot x_3(k+1) - \left(\alpha + \beta + \frac{\gamma}{4} - 3\right)\cdot y_{k+2} - \left(-2\cdot\alpha - \beta + \frac{\gamma}{4} + 3\right)\cdot y_{k+1} - (\alpha - 1)\cdot y_k$$

Let ky_αβϒ_delay = ky_αβϒ_delay(ω ~ 0) = 3

$$\mu_{y\_\alpha\beta\gamma\_C} := \frac{1}{1400}\cdot\sum_{k=501}^{1900}\left(y_k - CUBC\left(k - k_{y\_\alpha\beta\gamma\_delay}\right)\right)$$

$$\mu_{y\_\alpha\beta\gamma\_C} = -3.71\times10^{-4}$$

$$\sigma_{y\_\alpha\beta\gamma\_C} := \sqrt{\frac{1}{1400}\cdot\sum_{k=501}^{1900}\left[\left(y_k - CUBC\left(k - k_{y\_\alpha\beta\gamma\_delay}\right)\right) - \mu_{y\_\alpha\beta\gamma\_C}\right]^2}$$

$$\sigma_{y\_\alpha\beta\gamma\_C} = 6.4062\times10^{-4}$$

$$20\cdot\log\left(\frac{\sigma_{y\_\alpha\beta\gamma\_C}}{\sigma_{var}}\right) = -6.38\,dB$$

The result is identical to the step with additive noise. This is true for all filters as the signal type does not affect noise reduction!

Let kv_αβϒ_delay = kv_αβϒ_delay(ω ~ 0) = 4

Velocity α-β-γ **Ramp Input**

$$b_{k+3} := \left[\beta \cdot x_1(k+2) + \left(-2\cdot\beta + \frac{\gamma}{2}\right)\cdot x_1(k+1) + \left(\beta - \frac{\gamma}{2}\right)\cdot x_1(k) - \left(\alpha + \beta + \frac{\gamma}{4} - 3\right)\cdot b_{k+2} - \left(-2\cdot\alpha - \beta + \frac{\gamma}{4} + 3\right)\cdot b_{k+1} - (\alpha - 1)\cdot b_k\right]$$

$$v_{k+3} := b_{k+3}\cdot\frac{1\cdot s}{T_s}$$

$$\mu_{v\_\alpha\beta\gamma\_R} := \frac{1}{1400}\cdot\sum_{k=501}^{1900} v_k \qquad \mu_{v\_\alpha\beta\gamma\_R} = 10.00026$$

$$\sigma_{v\_\alpha\beta\gamma\_R} := \sqrt{\frac{1}{1400}\cdot\sum_{k=501}^{1900}\left(v_k - \mu_{v\_\alpha\beta\gamma\_R}\right)^2}$$

$$\sigma_{v\_\alpha\beta\gamma\_R} = 0.0436$$

Velocity α-β-γ **Parabolic Input**

$$b_{k+3} := \left[\beta \cdot x_2(k+2) + \left(-2\cdot\beta + \frac{\gamma}{2}\right)\cdot x_2(k+1) + \left(\beta - \frac{\gamma}{2}\right)\cdot x_2(k) - \left(\alpha + \beta + \frac{\gamma}{4} - 3\right)\cdot\left(b_{k+2}\right) - \left(-2\cdot\alpha - \beta + \frac{\gamma}{4} + 3\right)\cdot b_{k+1} - (\alpha - 1)\cdot b_k\right]$$

$$v_{k+3} := b_{k+3}\cdot\frac{1\cdot s}{T_s}$$

$$\mu_{v\_\alpha\beta\gamma\_P} := \frac{1}{1400}\cdot\sum_{k=501}^{1900}\left[v_k - A\cdot s\cdot\left(k - k_{v\_\alpha\beta\gamma\_delay}\right)\cdot T_s\right]$$

$$\mu_{v\_\alpha\beta\gamma\_P} = 2.55897\times 10^{-4}$$

$$\sigma_{v\_\alpha\beta\gamma\_P} := \sqrt{\frac{1}{1400}\cdot\sum_{k=501}^{1900}\left[\left[v_k - A\cdot s\cdot\left(k - k_{v\_\alpha\beta\gamma\_delay}\right)\cdot T_s\right] - \mu_{v\_\alpha\beta\gamma\_P}\right]^2}$$

$$\sigma_{v\_\alpha\beta\gamma\_P} = 0.0436$$

As before the noise reduction remains the same, as it does for any other input signal type.

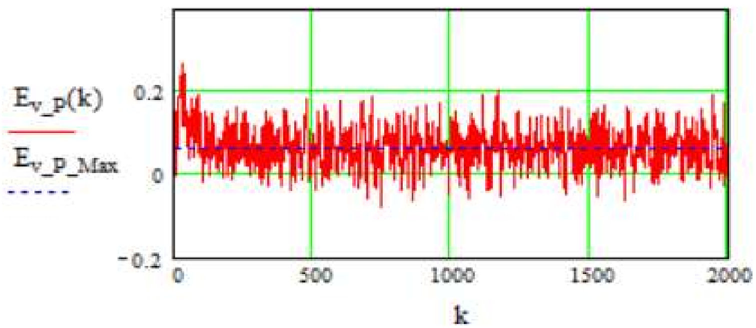
Note that A is multiplied by s times Ts, this makes the result unitless (as is v_k). It was my choice to do this. In reality the dimension is units / s.

Ramp of Velocity Output (due to parbolic input position)
Error Caused by Delay

$$E_{v\_P}(k) := A \cdot (k \cdot T_s) - v_k \cdot \frac{rad}{s}$$

$$E_{v\_P\_Max} := k_{v\_\alpha\beta\gamma\_delay} \cdot A \cdot T_s$$ **Maximum error for large k** **WO noise**

$$E_{v\_P\_Max} = 0.064 \frac{units}{s}$$



αβϒ Velocity Noise Output with offset due to Delay
Figure 16

Let ka_αβϒ_delay = ka_αβϒ_delay(ω ~ 0) = 53

Acceleration α-β-γ **Parabolic Input**

$$c_{k+3} := \frac{\gamma}{2} \cdot x_2(k+3) - \gamma \cdot x_2(k+2) + \frac{\gamma}{2} \cdot x_2(k+1) - \left(\alpha + \beta + \frac{\gamma}{4} - 3\right) \cdot c_{k+2} - \left(-2 \cdot \alpha - \beta + \frac{\gamma}{4} + 3\right) \cdot c_{k+1} - (\alpha - 1) \cdot c_k$$

$$a_{k+3} := c_{k+3} \cdot \left(\frac{1 \cdot s}{T_s}\right)^2$$

$$\mu_{a\_\alpha\beta\gamma\_P} := \frac{1}{1400} \cdot \sum_{k=:}^{1900} a_k$$

$$\mu_{a\_\alpha\beta\gamma\_P} = 10.00008$$

$$\sigma_{a\_\alpha\beta\gamma\_P} := \sqrt{\frac{1}{1400} \cdot \sum_{k=501}^{1900} \left(a_k - \mu_{a\_\alpha\beta\gamma\_P}\right)^2}$$

$$\sigma_{a\_\alpha\beta\gamma\_P} = 0.5426$$

Acceleration α-β-γ **CUBIC Input**

$$c_{k+3} := \frac{\gamma}{2}\cdot x_3(k+3) - \gamma\cdot x_3(k+2) + \frac{\gamma}{2}\cdot x_3(k+1) - \left(\alpha + \beta + \frac{\gamma}{4} - 3\right)\cdot c_{k+2} - \left(-2\cdot\alpha - \beta + \frac{\gamma}{4} + 3\right)\cdot c_{k+1} - (\alpha - 1)\cdot c_k$$

$$a_{k+3} := c_{k+3}\cdot\left(\frac{1\cdot s}{T_s}\right)^2$$

$$\mu_{a\_\alpha\beta\gamma\_C} := \frac{1}{1400}\cdot\sum_{k=501}^{1900}\left[a_k - J\cdot s^2\cdot\left(k - k_{a\_\alpha\beta\gamma\_delay}\right)\cdot T_s\right]$$

$$\mu_{a\_\alpha\beta\gamma\_C} = 6.4918\times 10^{-4}$$

$$\sigma_{a\_\alpha\beta\gamma\_C} := \sqrt{\frac{1}{1400}\cdot\sum_{k=501}^{1900}\left[\left[a_k - J\cdot s^2\cdot\left(k - k_{a\_\alpha\beta\gamma\_delay}\right)\cdot T_s\right] - \mu_{a\_\alpha\beta\gamma\_C}\right]^2}$$

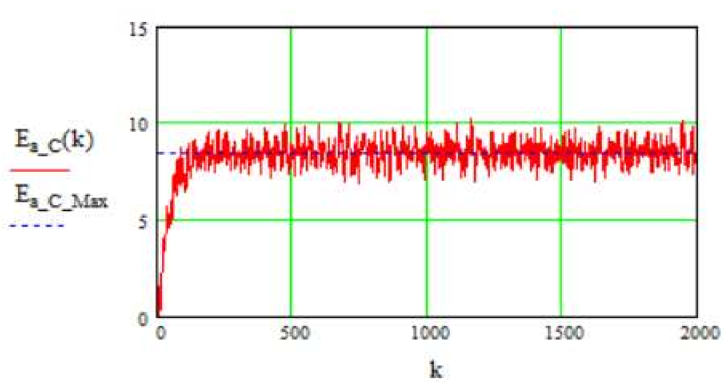
$$\sigma_{a\_\alpha\beta\gamma\_C} = 0.5426$$

Ramp of Acceleration Output (due to a cubic position input)
Error Caused by Delay

$$E_{a\_C}(k) := J\cdot\left(k\cdot T_s\right) - a_k\cdot\frac{rad}{s^2}$$

$$E_{a\_C\_Max} := k_{a\_\alpha\beta\gamma\_delay}\cdot J\cdot T_s$$ **WO Noise**

$$E_{a\_C\_Max} = 8.48\,\frac{rad}{s^2}$$ **Error is constant for large k**



αβϒ Acceleration Noise Output with offset due to Delay
Figure 17

Let ky_SG_delay = 3

Position SG **Step Input**

$$ysG_{k+2} := \frac{1}{35}\cdot\left(-3\cdot x_0(k) + 12\cdot x_0(k+1) + 17\cdot x_0(k+2) + 12\cdot x_0(k+3) - 3\cdot x_0(k+4)\right)$$

$$\mu_{y\_SG\_S} := \frac{1}{1400}\cdot\sum_{k=501}^{1900}\left(ysG_k - Y\right)$$

$$\mu_{y\_SG\_S} = 3.289\times10^{-5}$$

$$\sigma_{y\_SG\_S} := \sqrt{\frac{1}{1400}\cdot\sum_{k=501}^{1900}\left[\left(ysG_k - Y\right) - \mu_{y\_SG\_S}\right]^2}$$

$$\sigma_{y\_SG\_S} = 9.2276\times10^{-4}$$

$$20\cdot\log\left(\frac{\sigma_{y\_SG\_S}}{\sigma_{var}}\right) = -3.208\,\text{dB}$$

Not as good noise reduction as an αβϒ position section.

Position SG **CUBIC Input**

$$ysG_{k+2} := \frac{1}{35}\cdot\left(-3\cdot x_3(k) + 12\cdot x_3(k+1) + 17\cdot x_3(k+2) + 12\cdot x_3(k+3) - 3\cdot x_3(k+4)\right)$$

$$\mu_{y\_SG\_C} := \frac{1}{1400}\cdot\sum_{k=501}^{1900}\left(ysG_k - \text{CUBC}\left(k - k_{y\_SG\_delay}\right)\right)$$

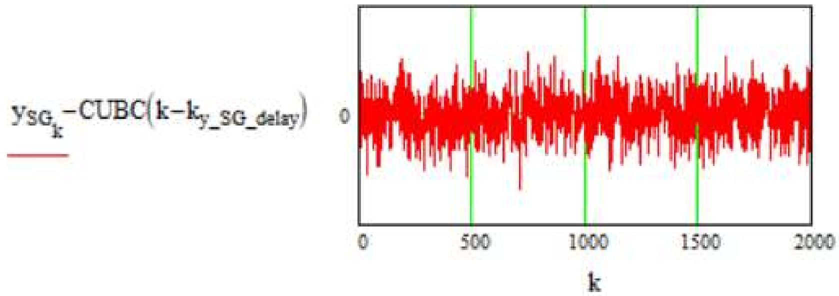
$$\mu_{y\_SG\_C} = 3.289\times10^{-5}$$

$$\sigma_{y\_SG\_C} := \sqrt{\frac{1}{1400}\cdot\sum_{k=501}^{1900}\left[\left(ysG_k - \text{CUBC}\left(k - k_{y\_SG\_delay}\right)\right) - \mu_{y\_SG\_C}\right]^2}$$

$$\sigma_{y\_SG\_C} = 9.2276\times10^{-4}$$

$$20\cdot\log\left(\frac{\sigma_{y\_SG\_C}}{\sigma_{var}}\right) = -3.208\,\text{dB}$$

As expected, the noise reduction is unchanged.



αβϒ Position Noise Output
Figure 18

A Summary of all noise calculations is presented in Table 1 below

Notice that the αβϒ acceleration section is shaded because of its limited bandwidth and time delay. We now proceed to a periodic signal noise analysis to show this.

Summary of Noise Calculations

Noise Comparisons

A = 0.300 **σvar = 1.3351E-03**
β = .0356 **Ts = 0.0016 s**
ϒ = 0.00141

Position	σy_αβϒ_i	**6.41E-04**		σy_SG_i	**9.23E-04**		σy_num_i	**7.80E-04**			
Velocity	σv_αβϒ_i	**0.044**		σv_SG_i	**0.262**		σv_num_i	**0.569**		σv_Rch_i	**0.766**
Acceleration	σa_αβϒ_i	**0.543**		σa_SG_i	**266.66**		σa_num_i	**1.29E+03**		σa_Rch_i	**1.63E+03**
RATIOs	σy_αβϒ_i / σvar	**0.480**		σy_SG_i /σvar	**0.691**		σy_num_i / σvar	**0.584**		σy_Rch_i / σvar	___
	σv_αβϒ_i / σvar	**32.96**		σv_SG_i /σvar	**196.24**		σv_num_i / σvar	**437.42**		σv_Rch_i /σvar	**573.74**
	σa_αβϒ_i / σvar	**406.71**		σa_SG_i /σvar	**2.00E+05**		σa_num_i / σvar	**9.66E+05**		σa_Rch_i /σvar	**1.22E+06**

Table 1.

Part 5– Periodic Noise Calculations – αβϒ with Sine Input

For this analysis the input is

$$x_4(k) := \left(x_{4\_init} + SIN(k-3) + on \cdot Var_k\right) \cdot \Phi(k-3)$$

where

$$SIN(k) := A_p \cdot \sin\left(k \cdot T_s \cdot \omega_{sig}\right) \cdot \Phi(k)$$

Let ωsig = 18 rad/s, Ap = 5 units and x4_int = 0

Using EQ 26, 27 and 28:

$$k_{y\_\alpha\beta\gamma\_delay}\left(18 \cdot \frac{rad}{s}\right) = 2.7855 \qquad k_{v\_\alpha\beta\gamma\_delay}\left(18 \cdot \frac{rad}{s}\right) = 9.7813 \qquad k_{a\_\alpha\beta\gamma\_delay}\left(18 \cdot \frac{rad}{s}\right) = 42.2468$$

From EQ 29, 30 and 31:

$$Ampl_y\left(\omega_{sig}\right) = 1.0138 \qquad Ampl_v\left(\omega_{sig}\right) = 1.1198 \qquad Ampl_a\left(\omega_{sig}\right) = 0.6387$$

$$A_{SIN}\left(\omega_0\right) := A_p \cdot Ampl_y\left(\omega_0\right) \qquad A_{SIN}\left(\omega_{sig}\right) = 2.0276$$ **Position**

$$A_{COS}\left(\omega_0\right) := \frac{\omega_0}{1 \cdot \frac{rad}{s}} \cdot A_p \cdot Ampl_v\left(\omega_0\right) \qquad A_{COS}\left(\omega_{sig}\right) = 40.3154$$ **Velocity**

$$A_{ISIN}\left(\omega_0\right) := \left(\frac{\omega_0}{1 \cdot \frac{rad}{s}}\right)^2 \cdot A_p \cdot Ampl_a\left(\omega_0\right) \qquad A_{ISIN}\left(\omega_{sig}\right) = 413.9618$$ **Acceleration**

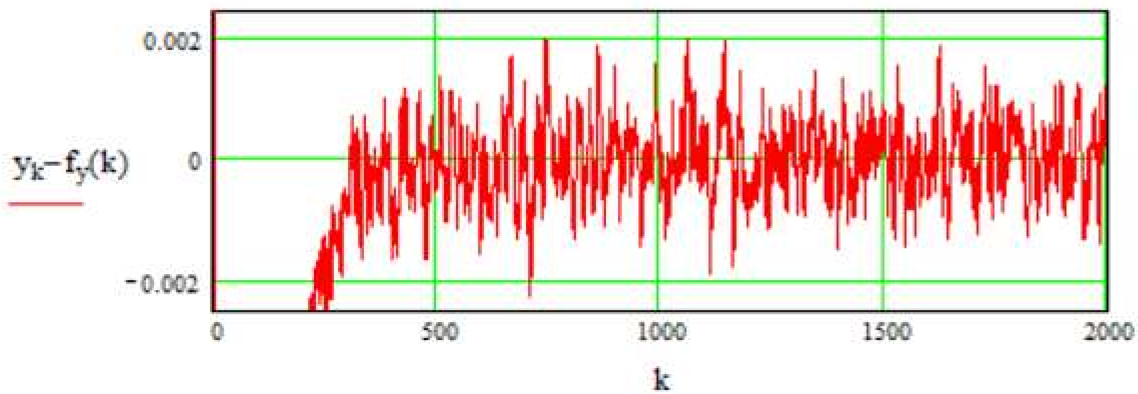
Let

$$f_y(k) := A_{SIN}(\omega_{sig})\cdot\sin\left[\omega_{sig}\cdot T_s\cdot\left(k - k_{y\_\alpha\beta\gamma\_delay}(\omega_{sig})\right)\right]$$

$$f_v(k) := A_{COS}(\omega_{sig})\cdot\cos\left[\omega_{sig}\cdot T_s\cdot\left(k - k_{v\_\alpha\beta\gamma\_delay}(\omega_{sig})\right)\right]$$

$$f_a(k) := A_{ISIN}(\omega_{sig})\cdot -\sin\left[\omega_{sig}\cdot T_s\cdot\left(k - k_{a\_\alpha\beta\gamma\_delay}(\omega_{sig})\right)\right]$$

Position α-β-γ **Sin Input**



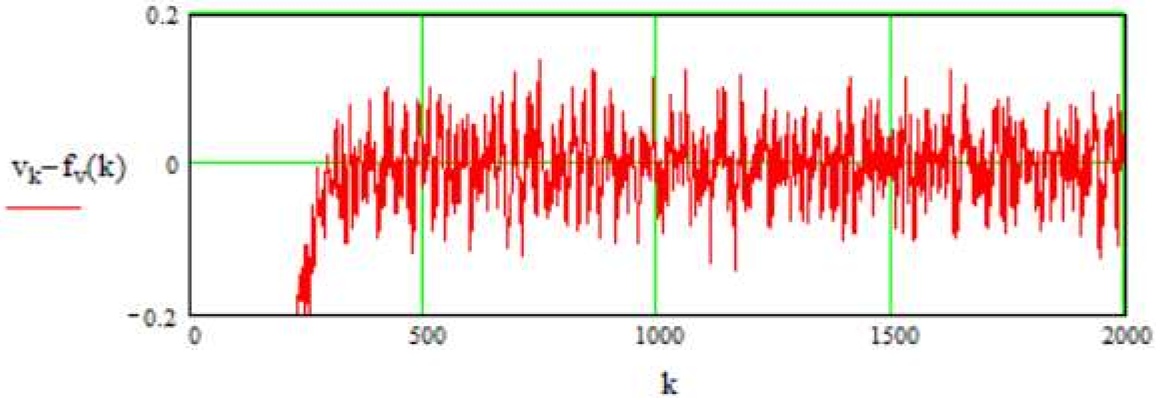
$$\mu_{y\_\alpha\beta\gamma\_Sin} := \frac{1}{1400}\cdot\sum_{k=501}^{1900}\left(y_k - f_y(k)\right)$$

$$\mu_{y\_\alpha\beta\gamma\_Sin} = 3.5611\times10^{-5}$$

$$\sigma_{y\_\alpha\beta\gamma\_Sin} := \sqrt{\frac{1}{1400}\cdot\sum_{k=501}^{1900}\left[\left(y_k - f_y(k)\right) - \mu_{y\_\alpha\beta\gamma\_Sin}\right]^2}$$

$$\sigma_{y\_\alpha\beta\gamma\_Sin} = 6.41\times10^{-4}$$

Velocity α-β-γ **SINE Input**



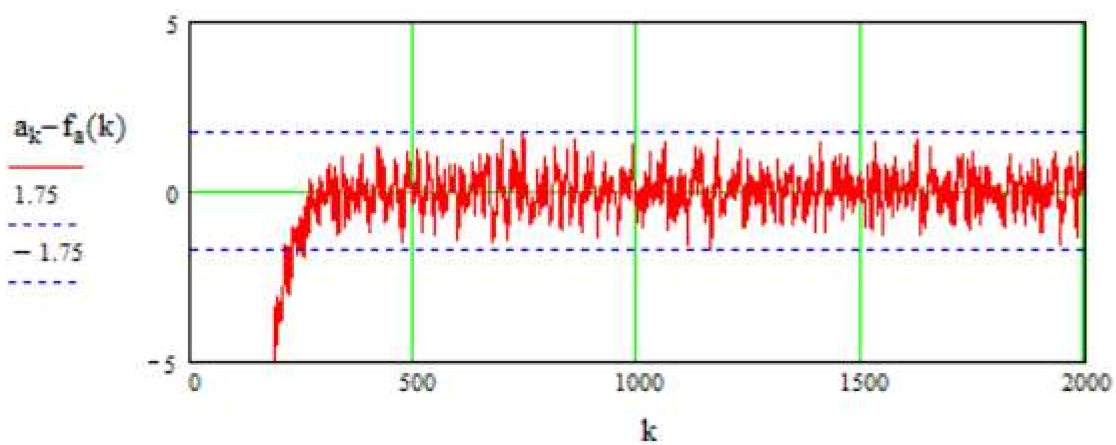
$$\mu_{v\_\alpha\beta\gamma\_Cos} := \frac{1}{1400} \cdot \sum_{k=501}^{1900} \left(v_k - f_v(k)\right)$$

$$\mu_{v\_\alpha\beta\gamma\_Cos} = 2.4864 \times 10^{-4}$$

$$\sigma_{v\_\alpha\beta\gamma\_Cos} := \sqrt{\frac{1}{1400} \cdot \sum_{k=501}^{1900} \left[\left(v_k - f_v(k)\right) - \mu_{v\_\alpha\beta\gamma\_Cos}\right]^2}$$

$$\sigma_{v\_\alpha\beta\gamma\_Cos} = 0.0436$$

Acceleration α-β-γ **SIN Input**



$$\mu_{a\_\alpha\beta\gamma\_ISin} := \frac{1}{1400} \cdot \sum_{k=501}^{1900} \left(a_k - f_a(k)\right)$$

$$\mu_{a\_\alpha\beta\gamma\_ISin} = 2.0667 \times 10^{-5}$$

$$\sigma_{a\_\alpha\beta\gamma\_ISin} := \sqrt{\frac{1}{1400} \cdot \sum_{k=501}^{1900} \left[\left(a_k - f_a(k)\right) - \mu_{a\_\alpha\beta\gamma\_ISin}\right]^2}$$

$$\sigma_{a\_\alpha\beta\gamma\_ISin} = 0.543$$

Again, these results match the aperiodic values.

Input Signal to Noise

$$20 \cdot \log\left(\frac{\frac{A_p}{\sqrt{2}}}{\sigma_{var}}\right) = 60.5\,\text{dB}$$

Output Signal to Noise – for each Filter Section

<u>**Incorporates Amplitude effects of filter type.**</u>

$$SNR_{y\_\alpha\beta\gamma} := 20 \cdot \log\left(\frac{\frac{A_{SIN}(\omega_{sig})}{\sqrt{2}}}{\sigma_{y\_\alpha\beta\gamma\_Sin}}\right) \qquad SNR_{y\_\alpha\beta\gamma} = 67\,\text{dB}$$

$$SNR_{v\_\alpha\beta\gamma} := 20 \cdot \log\left(\frac{\frac{A_{COS}(\omega_{sig})}{\sqrt{2}}}{\sigma_{v\_\alpha\beta\gamma\_Cos}}\right) \qquad SNR_{v\_\alpha\beta\gamma} = 56.32\,\text{dB}$$

$$SNR_{a\_\alpha\beta\gamma} := 20 \cdot \log\left(\frac{\frac{A_{ISIN}(\omega_{sig})}{\sqrt{2}}}{\sigma_{a\_\alpha\beta\gamma\_ISin}}\right) \qquad SNR_{a\_\alpha\beta\gamma} = 54.64\,\text{dB}$$

Relative Noise Levels

$$20 \cdot \log\left(\frac{\sigma_{y\_\alpha\beta\gamma\_Sin}}{\sigma_{var}}\right) = -6.378\,dB$$

$$20 \cdot \log\left(\frac{\sigma_{v\_\alpha\beta\gamma\_Cos}}{\sigma_{var}}\right) = 30.272\,dB$$

$$20 \cdot \log\left(\frac{\sigma_{a\_\alpha\beta\gamma\_ISin}}{\sigma_{var}}\right) = 52.18\,dB$$

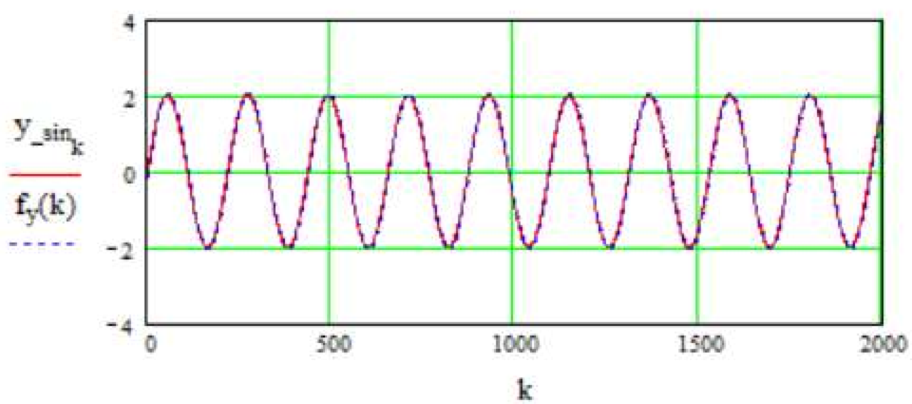
Recall that the first and second derivatives multiply the sinusoidal outputs by ωsig and ωsig2.
Next is shown a comparison between each ideal output and actual filter section output without additive noise:

Ideal Outputs for each filter section:

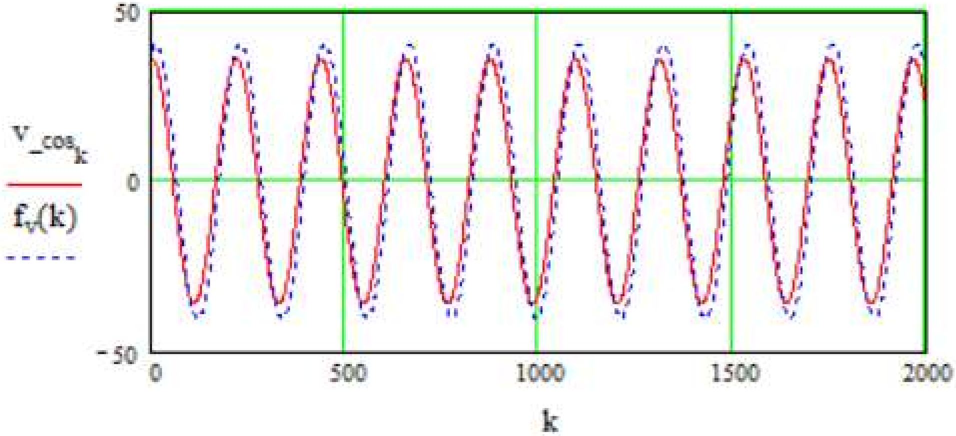
$$y\_sin_k := A_p \cdot \sin(\omega_{sig} \cdot T_s \cdot k)$$

$$v\_cos_k := A_p \cdot \omega_{sig} \cdot \cos(\omega_{sig} \cdot T_s \cdot k)$$

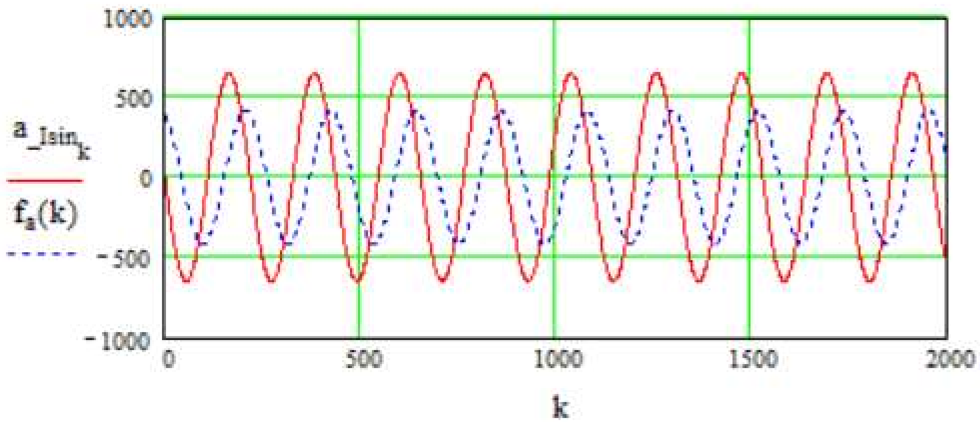
$$a\_Isin_k := A_p \cdot \omega_{sig}^{\;2} \cdot -\sin(\omega_{sig} \cdot T_s \cdot k)$$



Position Output Comparison - Ideal vs. Actual Output
Figure 19



Velocity Output Comparison - Ideal vs. Actual Output
Figure 20



Acceleration Output Comparison - Ideal vs. Actual Output
Figure 21

Part 6 – Bandwidth and Noise Dependance on αβϒ Parameters
Ts is constant = 0.0016 s

		Bandwidth and Noise vs. αβϒ Parameters Noise Input = **1.3351 E-03** **3dB Bandwidth** **BW / Ts**				Ts =1.6E-3s				
Item (n)	**Parameter**	**Pos BW**	**Noise Out**	**Ratio Noise**	**Vel BW**	**Noise Out**	**Ratio Noise**	**Accl BW**	**Noise Out**	**Ratio Noise**
0	α = 0.3	**0.44**	**6.41E-04**	1	**0.1784**	**0.0436**	1	**0.02396**	**0.5426**	1
1	α = 0.4	0.6184	7.44E-04	1.158	0.2548	0.0733	1.68	0.03429	1.3247	2.441
2	α = 0.5	0.8244	8.38E-04	1.307	0.3454	0.1133	2.599	0.04644	2.76	5.087
3	α = 0.625	1.1455	9.50E-04	1.48	0.4893	0.1862	4.2706	0.06556	6.43	11.85
4	α = 0.75	1.615	1.064E-03	1.66	0.6967	0.3038	7.2546	0.09205	14.85	27.4

Table 2

Let the Item Number = n and use it as a reference to point to the alpha entries. Note: The full αβϒ parameter set is referenced by α and/or n as tabulated below:

$n = 0$	$\alpha := 0.300$	$\beta := 0.0356$	$\gamma := 0.00141$
$n = 1$	$\alpha := 0.4$	$\beta := 0.06778$	$\gamma := 0.00384$
$n = 2$	$\alpha := 0.5$	$\beta := 0.11451$	$\gamma := 0.00878$
$n = 3$	$\alpha := 0.625$	$\beta := 0.20078$	$\gamma := 0.0217$
$n = 4$	$\alpha := 0.75$	$\beta := 0.33478$	$\gamma := 0.05067$

The β and ϒ values are derived from the document *Alpha-Beta-Gamma Tracking Filter Analysis* by this author. See EQ 38 and 39 on p 11. Analytical Reference is provided.

Bandwidth and noise patterns observed:

From Table 2, for **all** filter sections, a pattern emerges for the 3 dB bandwidth (the number which is divided by Ts, see blue headings). One sees that starting with n = 0, each successive entry is a multiple of the square root of 2. That is within +/- 10 %: *Valid over the range 0.3 ≤ α ≤ 0.75*

Define - ωy_αβϒ_3dB_0, ωv_αβϒ_3dB_0 and ωa_αβϒ_dB_0 as the initial values (n = 0).

Then -

$$\omega_{y\_\alpha\beta\gamma\_3dB}(n) := \omega_{y\_\alpha\beta\gamma\_3dB\_0}\cdot\left(\sqrt{2}\right)^{n} \qquad \textbf{EQ 32}$$

$$\omega_{v\_\alpha\beta\gamma\_3dB}(n) := \omega_{v\_\alpha\beta\gamma\_3dB\_0}\cdot\left(\sqrt{2}\right)^{n} \qquad \textbf{EQ 33}$$

$$\omega_{a\_\alpha\beta\gamma\_3dB}(n) := \omega_{a\_\alpha\beta\gamma\_3dB\_0}\cdot\left(\sqrt{2}\right)^{n} \qquad \textbf{EQ 34}$$

Where from Table 2 –

$$\omega_{y\_\alpha\beta\gamma\_3dB\_0} := 0.44$$

$$\omega_{v\_\alpha\beta\gamma\_3dB\_0} := 0.1784$$

$$\omega_{a\_\alpha\beta\gamma\_3dB\_0} := 0.02396$$

Next it is found that for each noise standard deviation ratio the following is –

For Position Section

$$\frac{\sigma_{y\_\alpha\beta\gamma\_i}(n)}{\sigma_{y\_\alpha\beta\gamma\_i\_0}} \sim \left(\frac{\omega_{y\_\alpha\beta\gamma\_3dB}(n)}{\omega_{y\_\alpha\beta\gamma\_3dB\_0}}\right)^{\frac{1}{2}} \qquad \textbf{EQ 35}$$

Velocity Section

$$\frac{\sigma_{v\_\alpha\beta\gamma\_i}(n)}{\sigma_{v\_\alpha\beta\gamma\_i\_0}} \sim \left(\frac{\omega_{v\_\alpha\beta\gamma\_3dB}(n)}{\omega_{v\_\alpha\beta\gamma\_3dB\_0}}\right)^{\frac{3}{2}} \qquad \textbf{EQ 36}$$

Acceleration Section

$$\frac{\sigma_{a\_\alpha\beta\gamma\_i}(n)}{\sigma_{a\_\alpha\beta\gamma\_i\_0}} \sim \left(\frac{\omega_{a\_\alpha\beta\gamma\_3dB}(n)}{\omega_{a\_\alpha\beta\gamma\_3dB\_0}}\right)^{\frac{5}{2}} \qquad \textbf{EQ 37}$$

Note that from Table 2

$$\sigma_{y\_\alpha\beta\gamma\_i\_0} = 6.41 \times 10^{-4}$$

$$\sigma_{v\_\alpha\beta\gamma\_i\_0} = 0.0436$$

$$\sigma_{a\_\alpha\beta\gamma\_i\_0} = 0.5426$$

Manipulating Equations 32 through 37 we find:

"Includes a small fudge factor." Accuracy is within +/- 5.5 % for any α or α(n).

$$\sigma_{y\_\alpha\beta\gamma\_i}(n) := \left[\sigma_{y\_\alpha\beta\gamma\_i\_0}\cdot\left(\sqrt{2}\right)^{\frac{n}{2}}\right]\cdot 0.96^{n} \qquad \textbf{EQ 38}$$

$$\sigma_{v\_\alpha\beta\gamma\_i}(n) := \left[\sigma_{v\_\alpha\beta\gamma\_i\_0}\cdot\left(\sqrt{2}\right)^{\frac{3\cdot n}{2}}\right]\cdot 0.97^{n} \qquad \textbf{EQ 39}$$

$$\sigma_{a\_\alpha\beta\gamma\_i}(n) := \left[\sigma_{a\_\alpha\beta\gamma\_i\_0}\cdot\left(\sqrt{2}\right)^{\left(\frac{5\cdot n}{2}\right)\cdot 0.9845^{n}}\right] \qquad \textbf{EQ 40}$$

Estimated Noise Output:

Position Noise Out $\sigma_{y\_\alpha\beta\gamma\_i}(n) =$	Velocity Noise Out $\sigma_{v\_\alpha\beta\gamma\_i}(n) =$	Acceleration Noise Out $\sigma_{a\_\alpha\beta\gamma\_i}(n) =$
$6.41\cdot10^{-4}$	0.0436	0.5426
$7.318\cdot10^{-4}$	0.0711	1.2733
$8.354\cdot10^{-4}$	0.116	2.9101
$9.538\cdot10^{-4}$	0.1893	6.4813
$1.089\cdot10^{-3}$	0.3088	14.0753

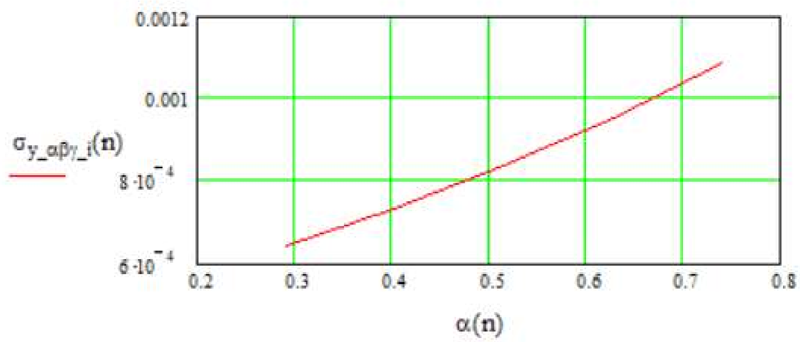
Values compare well with those in Table 2.

Note that the power in EQ 40 is **(5 n / 2) 0.9845^{n}.**

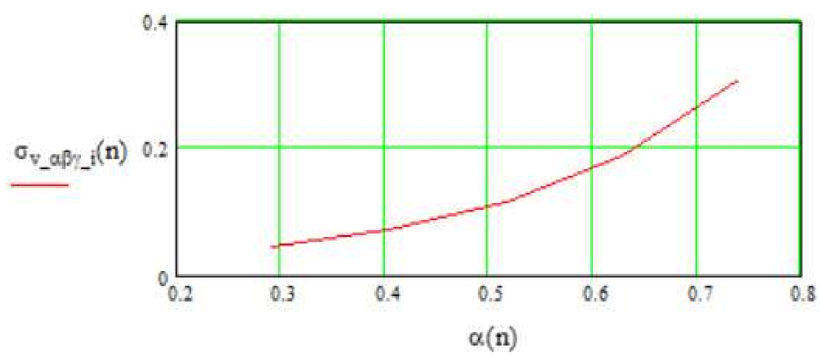
Below are plots of Noise Output vs α for each αβϒ Filter Section:

Alpha conversion from n

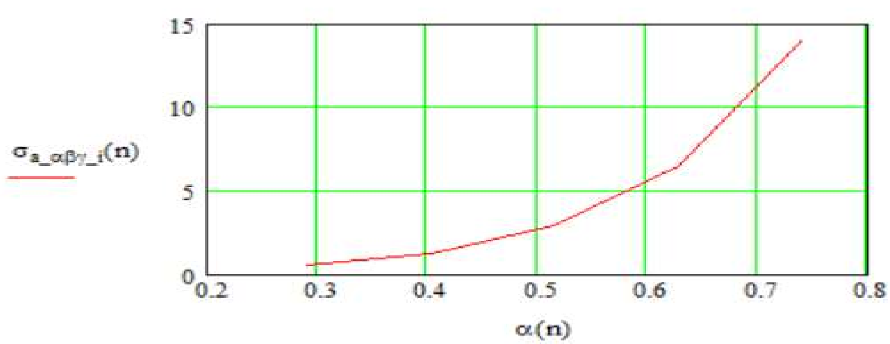
$$\alpha(n) := 0.1125 \cdot n + 0.29$$



Position Section Noise Out as a Function of α(n)
Figure 22



Velocity Section Noise Out as a Function of α(n)
Figure 23



Acceleration Section Noise Out as a Function of α(n)
Figure 24

Part 7 – Observations and Conclusions

Observations:

1. For **all** position filter sections – output noise does <u>not</u> depend on the sampling rate.

2. For **all** velocity filter sections – output noise is inversely proportional to the sampling rate.

$$\sigma_{v\_} \propto \frac{1}{T_s}$$

3. For **all** acceleration filter sections – output noise is inversely proportional to the sampling rate squared.

$$\sigma_{a\_} \propto \frac{1}{T_s^{\,2}}$$

   So Ts should be as large as possible but is bound by the Nyquist Rate Criterion of:

   Let f_{max} represent the largest frequency present (bandwidth)

   Sampling Rate = $f_s = \frac{1}{T_s} = \frac{\omega_s}{2\cdot\pi}$

   Nyquist Rate = $2\cdot f_{max}$

$$\frac{1}{2\cdot T_s} \geq f_{max} = \frac{\omega_{max}}{2\cdot\pi}$$

4. The αβϒ filter provides the best overall noise reduction of all filters. However, being an Infinite Impulse Response (IIR) type digital filter, there is a non-linear phase response (Refer to Figures 1 b, 2 b and 3 b) and therefore the time delay is a function of frequency. This is captured in Figures 12, 13 and 14.

5. For the αβϒ Filter the lower the α the better the noise performance (because of lower bandwidth). Refer to Figures 22, 23 and 24.

6. The SG filter is the best filter choice when processing sine-cosine encoder outputs. This is due to the linear phase of all filter sections allowing higher sinusoidal frequencies without distortion. Refer below to Reference 13.

7. Equations EQ 26 through 31 are valid for any α value!

Part 8 – References

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[2] Tenne, D. & Singh, T. (2000), *Optimal Design of α-β-(ϒ) Filters,* Center for Multisource Information Fusion, State University of New York at Buffalo, Buffalo, NY 14269, Pages 1-5. http://code.eng.buffalo.edu/tracking/papers/tenne_optimalFilter2000.pdf Published in American Control Conference 2000 – Engineering, DOI 10.1109/ACC.2000.877043

[3] Henry Stern (2021), *Analytical Expressions – From Trend-Following Filters – Part 1/2* Research Insights, Factor Investing, Trend Following Trend-Following Filters: Part 1/2 - (alphaarchitect.com) **Section 6**. Alpha-Beta (α-β) Tracking Filter.

[4] Henry Stern (2021), *Analytical Expressions – From Trend-Following Filters – Part 2/2* Research Insights, Factor Investing, Trend Following Trend-Following Filters – Part 2/2 - (alphaarchitect.com) **Section 5.** Alpha-Beta-Gamma (α−β−γ) Tracking Filter.

[5] Dirk Tenne and Tarunraj Singh (2001), *Optimal Design of α-β-(ϒ) Filters,* Center for Multisource Information Fusion, State University of New York at Buffalo, Buffalo, NY 14269, Pages 1-5.

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{8) Savitzky–golay filter. (2023, September 23). In *Wikipedia*.

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[11] Doran Levy, (2017), *Introduction to Numerical Analysis*, Department of Mathematics and Center for Computation and Modeling (CSCAAMM), University of Maryland pp 1-3 and pp 8-10.

[12] Finite Difference Coefficients Calculator

```
                @misc{fdcc,
  title={Finite Difference Coefficients Calculator},
  author={Taylor, Cameron R.},
  year={2016},
  howpublished="\url{https://web.media.mit.edu/~crtaylor/calculator.html}"
}
```

[13] Manuel Alcázar Vargas, Javier Pérez Fernández, Juan M. Velasco García, Juan A. Cabrera Carrillo, and Juan J. Castillo Aguilar, Sensors (Basel). 2021 Jan; 21(2): 577. Published online 2021 Jan15. doi: 10.3390/s21020577 *A Novel Method for Determining Angular Speed and Acceleration Using Sin-Cos Encoders*