Cascade Velocity PI Controller Torque Disturbance Analysis

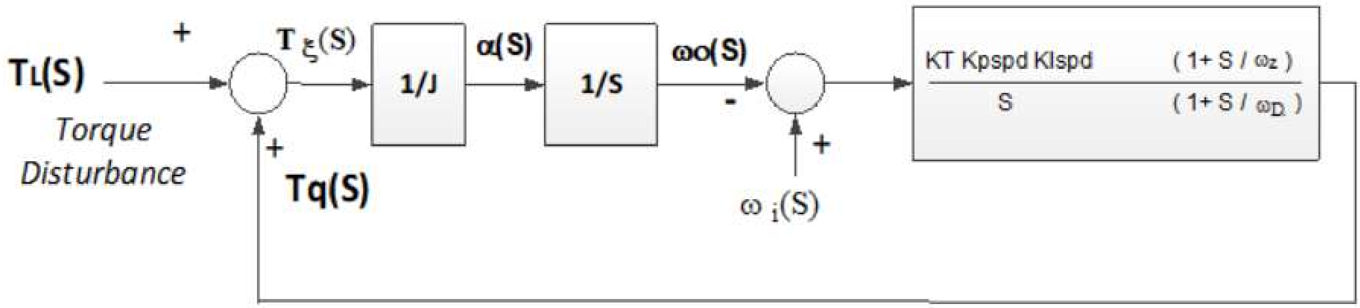


Figure 1
Velocity Loop Model with a Torque Disturbance

The intent of this document is to provide a Mathcad model which can be used to evaluate the transient response to various torque disturbances applied at the load.

Note that the model does not include a prefilter. A prefilter does not impact the response to a torque disturbance for a velocity loop. In a position loop implementation the outer loop is much slower than the inner velocity loop.

We divide the analysis into 3 zones.

1. $X = 3$
2. $2 \le \chi < 3$
3. $3 < \chi < 25$

This was done to provide the best matching of estimates to the peak amplitude and its time of occurrence. These estimates were provided in the document *Cascade PI Controller Design using a Limited Span Method* (LSM) by EQ 33 through EQ 36.

The time response was adjusted to each zone as a function of the damping factor ζ.

Background:

$T_{\xi}(S) = J \cdot \alpha(S) + D \cdot \omega(S)$

J (kg m^2) is the total moment of inertia.

$\alpha = \frac{d}{dt}\omega(t) \qquad \frac{\left(\frac{rad}{s}\right)}{s}$ **Angular Acceleration**

$D \qquad \frac{N \cdot m}{\left(\frac{rad}{s}\right)}$ **Viscus Friction constant.**

If we can neglect the viscus friction term -

$$\alpha(S) = \frac{T_{\xi}(S)}{J} \qquad \left[\frac{\left(\frac{rad}{s}\right)}{s}\right] \qquad \frac{N \cdot m}{kg \cdot m^2} = 1 \frac{\left(\frac{rad}{s}\right)}{s}$$

As we desire the outout angular speed change with respect to a load torque disturbance:

$$\omega(S) = \frac{T_{\xi}(S)}{S \cdot J} \qquad \left(\frac{rad}{s}\right)$$

In the time domain - $\omega(t) = \int \frac{1}{J} \cdot T_{\xi}(t)\,dt$

Conversion Factors

$ounce\_in := \frac{1 \cdot N \cdot m}{141.61193} \qquad Rev := 2 \cdot \pi \cdot rad \qquad RPM := 1 \cdot \frac{Rev}{60 \cdot s}$

Defined Inputs

$\chi := 1$ placeholder

$J := 2.5 \cdot 10^{-5} \cdot kg \cdot m^2$ Total Moment of Inertia $\qquad \omega_D := 100 \cdot \frac{rad}{s}$

$T_{Lp} := 7.0611519 \cdot 10^{-3} \cdot N \cdot m$ Amplitude of the Torque Disturbance

$T_{Lp} = 1\ ounce\_in$

From the Model

$$T_\xi(S) = T_L(S) + T_q(S)$$ **Summing Point for the torque disturbance**

With an Open Loop Gain

$$G_{ol\_spd}(S) = \frac{1}{J \cdot S} \cdot \frac{K_T \cdot K_{pspd} \cdot K_{Ispd}}{S} \cdot \frac{\left(1 + \frac{S}{\omega_z}\right)}{\left(1 + \frac{S}{\omega_D}\right)}$$ **EQ 1**

$$T_q(S) = -G_{ol\_spd}(S) \cdot T_\xi(S) = -G_{ol\_spd}(S) \cdot \left(T_L(S) + T_q(S)\right)$$

$$T_q(S) \cdot \left(1 + G_{ol\_spd}(S)\right) = -G_{ol\_spd}(S) \cdot T_L(S)$$

$$T_q(S) = \frac{-G_{ol\_spd}(S)}{1 + G_{ol\_spd}(S)} \cdot T_L(S)$$ **EQ 2**

Therefore $$T_\xi(S) = T_L(S) - \frac{G_{ol\_spd}(S)}{1 + G_{ol\_spd}(S)} \cdot T_L(S)$$ **EQ 3**

$$T_\xi(S) = \frac{\left(1 + G_{ol\_spd}(S)\right) - G_{ol\_spd}(S)}{1 + G_{ol\_spd}(S)} \cdot T_L(S)$$

$$T_\xi(S) = \frac{1}{1 + G_{ol\_spd}(S)} \cdot T_L(S)$$

Note that the Sensitivity Function is $$S_{\xi\_spd}(S) = \frac{1}{1 + G_{ol\_spd}(S)}$$ **EQ 4**

$$\text{Therefore } \; T_\xi(S) = S_{\xi\_spd}(S) \cdot T_L(S)$$ **EQ 5**

$$S_{\xi\_spd}(S) := \frac{S^2 \cdot (S + \omega_D)}{S^3 + S^2 \cdot \omega_D + S \cdot \frac{\omega_D^{\,2}}{\chi} + \frac{\omega_D^{\,3}}{\chi^3}} \qquad \textbf{EQ 6}$$

$$\frac{\omega_{out}(S)}{T_L(S)} = \frac{1}{S \cdot J} \cdot S_{\xi\_spd}(S) \qquad \textbf{EQ 7}$$

In all cases the poles and zeroes are established by equations listed in the document *Cascade PI Controller Design using a Limited Span Method* – Closed Loop Summary (pages 10 and 11)

Throughout

$J := 2.5 \cdot 10^{-5} \cdot kg \cdot m^2 \qquad T_{Lp} := 7.0611519 \cdot 10^{-3} \cdot N \cdot m$

See **Page 11**, χ = 3, ζ = 1 in the document: *Cascade PI Controller Design using a Limited Span Method*.

Zone 1 $\chi := 3$

$t := 0 \cdot s, 0.0001 \cdot s .. 0.8 \cdot s$

$\sigma(\chi) := \frac{\omega_D}{2\chi}(\chi - 1) \qquad \sigma(\chi) = 33.333 \frac{rad}{s}$

$\omega_1(\chi) := \sigma(\chi) \cdot \left[1 - j \cdot \frac{\sqrt{-\chi^2 + 2 \cdot \chi + 3}}{(\chi - 1)} \right] \qquad \omega_1(\chi) = 33.333 \frac{rad}{s}$

$\omega_2(\chi) := \sigma(\chi) \cdot \left[1 + j \cdot \frac{\sqrt{-\chi^2 + 2 \cdot \chi + 3}}{(\chi - 1)} \right] \qquad \omega_2(\chi) = 33.333 \frac{rad}{s}$

$\omega_c(\chi) := \frac{\omega_D}{\chi} \qquad \omega_c(\chi) = 33.333 \frac{rad}{s}$

$K_A(\chi) := \frac{\omega_D^2}{\chi^3} \qquad \omega_z(\chi) := \frac{\omega_D}{\chi^2}$

$K_A(\chi) = 370.37 \frac{rad^2}{s^2} \qquad \omega_z(\chi) = 11.111 \frac{rad}{s}$

$\omega_n := \omega_c(\chi) \qquad = \qquad \sqrt{\omega_1(\chi) \cdot \omega_2(\chi)} = 33.333 \frac{rad}{s} \qquad \zeta := \frac{\sigma(\chi)}{\omega_n} \qquad \zeta = 1$

See **Page 21**, EQ 33, and EQ 35 in the document: *Cascade PI Controller Design using a Limited Span Method.*

FOR $\chi = 3$ ***Estimation***

$$T_{peak\_L}(\chi,\omega_D) := \frac{\left(0.2163\cdot\chi^2 + 0.7558\cdot\chi + 0.6984\right)}{\omega_D}$$

$$a := -0.35887268 \qquad \omega_{ref\_L} := 0.74967119\cdot\frac{rad}{s}$$

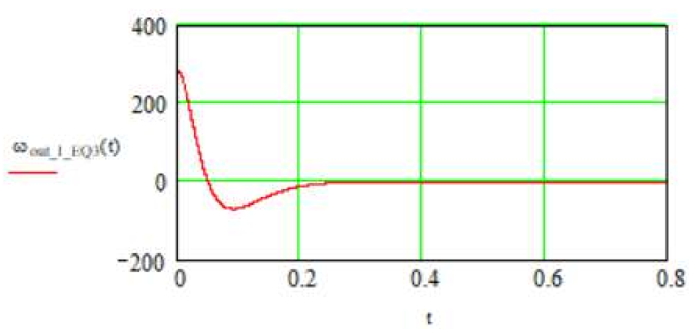
$$\Delta\omega_{o\_peak\_L}(\chi,\omega_D) := \left[(\chi - a)\cdot\left(\frac{1\cdot\frac{rad}{s}}{\omega_D}\cdot\frac{T_{Lp}}{1\cdot N\cdot m}\cdot\frac{1\cdot kg\cdot m^2}{J}\right)\right]\cdot\omega_{ref\_L}$$

$$T_{peak\_L}(3,\omega_D) = 0.049\,s \qquad \Delta\omega_{o\_peak\_L}(3,\omega_D) = 7.112\,\frac{rad}{s}$$

Time Domain response to a Impulse of torque

$$\frac{S\cdot(S+\omega_D)}{\left(S+\omega_c(\chi)\right)^3}\cdot\frac{T_{Lp}}{J}\cdot\frac{kg\cdot m^2}{N\cdot m}\cdot\frac{1}{s}$$ **EQ 8** χ **= 3 [ζ = 1]**

$$\omega_{out\_I\_EQ3}(t) := \frac{T_{Lp}}{J}\cdot\frac{kg\cdot m}{N}\cdot\frac{1}{s}\cdot\left(\frac{-1}{2}\cdot t^2\cdot\omega_c(\chi)\cdot\omega_D + \frac{1}{2}\cdot\omega_c(\chi)^2\cdot t^2 + t\cdot\omega_D - 2\cdot t\cdot\omega_c(\chi) + 1\right)\cdot\exp\left(-\omega_c(\chi)\cdot t\right)$$ **EQ 9**

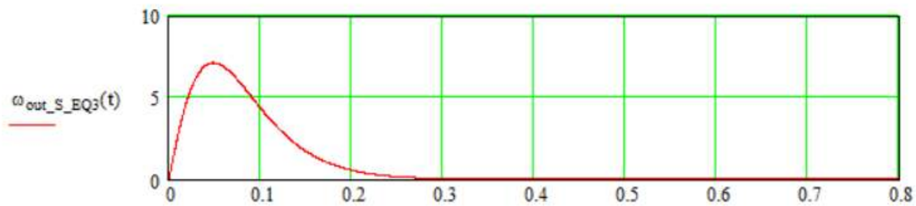


$$\omega_{out\_I\_EQ3}(0\cdot s) = 282.446\,\frac{rad}{s}$$

$$\frac{T_{Lp}}{J} = 282.446\,\frac{rad}{s^2}$$

Time Domain Response to a Step of Torque

$$\omega_{out\_S\_EQ3}(t) := -\left[\frac{\frac{1}{2}\cdot\exp(-t\cdot\omega_c(\chi))\cdot t\cdot\omega_c(\chi)\cdot(-\omega_D\cdot t + t\cdot\omega_c(\chi) - 2)}{\omega_c(\chi)}\right]\cdot\left(\frac{T_{Lp}}{J}\cdot\frac{kg}{N\cdot m}\cdot m^2\cdot\frac{1}{s^2}\right) \quad \left(\frac{rad}{s}\right) \qquad \textbf{EQ 10}$$



$$\omega_{out\_S\_EQ3}(0.049\cdot s) = 7.117\,\frac{rad}{s} \qquad \text{Peak}$$

$$\omega_{out\_S\_EQ3}(0.2806\cdot s) = 0.071\,\frac{rad}{s} \qquad \text{1\% of Peak}$$

$t_v := 0.03\cdot s$

$T_v := \text{Maximize}(\omega_{out\_S\_EQ3}, t_v)$

Estimated		**Actual**
$\Delta\omega_{o\_peak\_L}(3,\omega_D) = 7.112\,\frac{rad}{s}$		$\omega_{out\_S\_EQ3}(T_v) = 7.11732\,\frac{rad}{s}$
	Both are zero if out of range	
$T_{peak\_L}\left(3, 100\cdot\frac{rad}{s}\right) = 0.049\,s$		$T_v = 0.049\,s$

$\frac{28.06}{\omega_D} = 0.2806\,s \qquad \omega_{out\_S\_EQ3}(0.2806\cdot s) = 0.0711\,Hz$ **Recovery to within ~ 1% of peak**

$$\frac{\frac{\Delta\omega_{o\_peak\_L}(3,\omega_D)}{100}}{\omega_{out\_S\_EQ3}(0.28060\cdot s)} = 0.999994$$

Note that at the end of the move due to the torque step disturbance a residual phase offset will result. This is the result of integrating the angular frequency plot (EQ 10 above, χ = 3.0).

$$\lim_{t_1\to\infty}\int_0^{t_1}\omega_{out\_S\_EQ3}(t)\,dt = 0.763\cdot rad$$

See **Page 21**, EQ 33, and EQ 35 in the document: *Cascade PI Controller Design using a Limited Span Method*.

Zone 2 **$2 <= \chi < 3$** $\omega := 0 \cdot \frac{rad}{s}, 0.1 \cdot \frac{rad}{s} .. 800 \cdot \frac{rad}{s}$

Example $\chi := 2.716$

$C := 1$ $D := 0$

$IS\_CHI\_OK23 := [if[(2 \le \chi < 3), C, D]]$

$IS\_CHI\_OK23 = 1$

O identifies $2 <= \chi < 3$

$O := [if[(\chi \ge 2) \cdot IS\_CHI\_OK23 \wedge (\chi < 3) \cdot IS\_CHI\_OK23, 1, 0]]$

$O = 1$ **O = 1 if in range, 0 otherwise**

See **Page 11**, $\chi < 3$, $\zeta < 1$ in the document: *Cascade PI Controller Design using a Limited Span Method*.

$$\sigma(\chi) := \frac{\omega_D}{2\chi}(\chi - 1)$$

$$\omega_1(\chi) := \sigma(\chi)\cdot\left[1 - j\cdot\frac{\sqrt{-\chi^2 + 2\cdot\chi + 3}}{(\chi - 1)}\right] \qquad \omega_1(\chi) = 31.591 - 18.912i\,\frac{rad}{s}$$

$$\omega_2(\chi) := \sigma(\chi)\cdot\left[1 + j\cdot\frac{\sqrt{-\chi^2 + 2\cdot\chi + 3}}{(\chi - 1)}\right] \qquad \omega_2(\chi) = 31.591 + 18.912i\,\frac{rad}{s}$$

$$\omega_c(\chi) := \frac{\omega_D}{\chi} \qquad \omega_c(\chi) = 36.819\,\frac{rad}{s}$$

$$K_A(\chi) := \frac{\omega_D^{\,2}}{\chi^3} \qquad \omega_z(\chi) := \frac{\omega_D}{\chi^2}$$

$$K_A(\chi) = 499.127\,\frac{rad^2}{s^2} \qquad \omega_z(\chi) = 13.556\,\frac{rad}{s}$$

$$\omega_n := \omega_c(\chi) \qquad = \sqrt{\omega_1(\chi)\cdot\omega_2(\chi)} = 36.819\,\frac{rad}{s} \qquad \zeta := \frac{\sigma(\chi)}{\omega_n} \qquad \zeta = 0.858$$

See **Page 21**, EQ 33, and EQ 35 in the document: *Cascade PI Controller Design using a Limited Span Method.*

$$\omega_c(\chi) = 36.819\,\frac{rad}{s} \qquad \alpha := \zeta\cdot\omega_n \qquad \omega_d := \omega_n\cdot\sqrt{1 - \zeta^2} \qquad \alpha^2 + \omega_d^{\,2} = \omega_n^{\,2}$$

$$\alpha = 31.591\,Hz \qquad \omega_d = 18.912\,Hz \qquad \zeta = 0.858$$

$$\sqrt{\alpha^2 + \omega_d^{\,2}} = 36.819\,Hz$$

$$a := -0.35887268 \qquad \omega_{ref\_L} := 0.74967119\cdot\frac{rad}{s}$$

$$b := 0.46092839 \qquad \omega_{ref\_U} := 0.91066589\cdot\frac{rad}{s}$$

$$T_{peak}(\chi, \omega_D) := \left[\text{if}\left[(O = 1), \frac{\left(0.2163\cdot\chi^2 + 0.7558\cdot\chi + 0.6984\right)}{\omega_D}, \frac{\left(0.0409\cdot\chi^2 + 2.4526\cdot\chi - 3.685\right)}{\omega_D}\right]\right]$$

$$\Delta\omega_o(\chi, \omega_D) := \left[\text{if}\left[(O = 1), (\chi - a)\cdot\left(\frac{1\cdot\frac{rad}{s}}{\omega_D}\cdot\frac{T_{Lp}}{1\cdot N\cdot m}\cdot\frac{1\cdot kg\cdot m^2}{J}\right)\cdot\omega_{ref\_L}, (\chi - b)\cdot\left(\frac{1\cdot\frac{rad}{s}}{\omega_D}\cdot\frac{T_{Lp}}{1\cdot N\cdot m}\cdot\frac{1\cdot kg\cdot m^2}{J}\right)\cdot\omega_{ref\_U}\right]\right]$$

Time domain response to a Impulse of torque

$t := 0 \cdot s, 0.0001 \cdot s .. 0.3 \cdot s$ Time Range Variable

$$\frac{S \cdot (S + \omega_D)}{\left[(S + \alpha)^2 + \omega_d^{\,2}\right]} \cdot \frac{1}{S + \omega_c(\chi)} \cdot \frac{T_{Lp}}{J} \cdot \frac{kg \cdot m^2}{N \cdot m} \cdot \frac{1}{s}$$ **EQ 11** **for $\zeta < 1$** **Impulse Response**

$\chi < 3$

Let

$$D_{en} := \alpha^2 + \omega_d^{\,2} + \omega_c(\chi)^2 - 2 \cdot \alpha \cdot \omega_c(\chi)$$ **EQ 12**

and

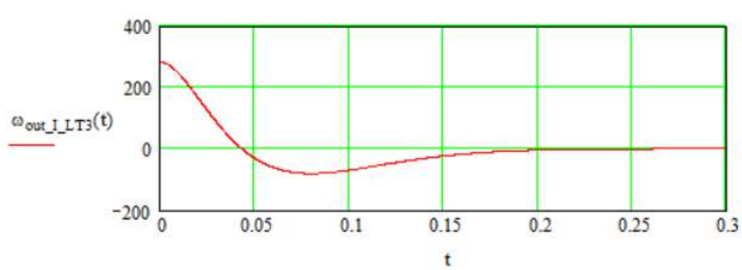
$$k_1(t) := \frac{T_{Lp}}{J} \cdot \left(\frac{kg \cdot m^2}{N \cdot m} \cdot 1 \cdot \frac{rad}{s} \right) \cdot \frac{1}{D_{en}} \cdot \left[\left(\omega_d^{\,2} + \alpha^2 + \omega_c(\chi) \cdot \omega_D - 2 \cdot \alpha \cdot \omega_c(\chi) \right) \cdot \exp(-\alpha \cdot t) \cdot \cos(\omega_d \cdot t) \right]$$ **EQ 13**

$$k_2(t) := \frac{T_{Lp}}{J} \cdot \left(\frac{kg \cdot m^2}{N \cdot m} \cdot 1 \cdot \frac{rad}{s} \right) \cdot \frac{1}{D_{en}} \cdot \left(-\omega_d \cdot \omega_c(\chi) - \alpha \cdot \omega_d + \omega_D \cdot \omega_d \right) \cdot \exp(-\alpha \cdot t) \cdot \sin(\omega_d \cdot t)$$ **EQ 14**

$$k_3(t) := \frac{T_{Lp}}{J} \cdot \left(\frac{kg \cdot m^2}{N \cdot m} \cdot 1 \cdot \frac{rad}{s} \right) \cdot \frac{1}{D_{en}} \cdot \left[\frac{-\alpha \cdot \left(\alpha^2 + \omega_D \cdot \omega_c(\chi) \right)}{\omega_d} + \frac{\alpha^2 \cdot \left(\omega_c(\chi) + \omega_D \right)}{\omega_d} \right] \cdot \exp(-\alpha \cdot t) \cdot \sin(\omega_d \cdot t)$$ **EQ 15**

$$k_4(t) := \frac{T_{Lp}}{J} \cdot \left(\frac{kg \cdot m^2}{N \cdot m} \cdot 1 \cdot \frac{rad}{s} \right) \left[\frac{1}{D_{en}} \cdot \left(\omega_c(\chi)^2 - \omega_D \cdot \omega_c(\chi) \right) \cdot \exp(-\omega_c(\chi) \cdot t) \right]$$ **EQ 16**

$$\omega_{out\_I\_LT3}(t) := k_1(t) + k_2(t) + k_3(t) + k_4(t)$$ **EQ 17**



Time domain response to a Step of torque

$$\frac{(S+\omega_D)}{\left[(S+\alpha)^2+\omega_d{}^2\right]}\cdot\frac{1}{(S+\omega_c(\chi))}$$ **EQ 18** **Step Response** **for $\zeta < 1$**

$\chi < 3$

Let

$$X_1 := \frac{\alpha}{\omega_n{}^2}$$ **EQ 19**

$$X_2 := \frac{\omega_d}{\omega_n{}^2}$$ **EQ 20**

and

$$K_{1A}(t) := \exp(-\alpha\cdot t)\cdot\left(-X_1\cdot\omega_d{}^2 + -X_1\cdot\alpha^2 + -X_2\cdot\omega_d\cdot\omega_D + X_2\cdot\omega_d\cdot\omega_c(\chi) + -X_1\cdot\omega_c(\chi)\cdot\omega_D\right)$$ **EQ 21**

$$K_{1B}(t) := \exp(-\alpha\cdot t)\cdot\left(X_2\cdot\omega_d{}^2 + X_2\cdot\alpha^2 + -X_1\cdot\omega_d\cdot\omega_D + X_1\cdot\omega_d\cdot\omega_c(\chi) + X_2\cdot\omega_c(\chi)\cdot\omega_D\right)$$ **EQ 22**

$$I_1(t) := \left(\frac{T_{Lp}}{J}\cdot\frac{kg\cdot m}{N}\cdot\frac{1}{D_{en}\cdot s^2}\right)\cdot\left[(K_{1A}(t))\cdot\cos(\omega_d\cdot t) + K_{1B}(t)\cdot\sin(\omega_d\cdot t)\right]$$ **EQ 23**

$$K_{2A}(t) := \exp(-\alpha\cdot t)\cdot\left(2\cdot X_1\cdot\alpha\cdot\omega_c(\chi) + -X_2\cdot\frac{\alpha^2\cdot\omega_D}{\omega_d} + X_2\cdot\frac{\alpha^3}{\omega_d} + -X_2\cdot\frac{\alpha^2\cdot\omega_c(\chi)}{\omega_d} + X_2\cdot\frac{\alpha\cdot\omega_c(\chi)\cdot\omega_D}{\omega_d}\right)$$ **EQ 24**

$$K_{2B}(t) := \exp(-\alpha\cdot t)\cdot\left(-2\cdot X_2\cdot\alpha\cdot\omega_c(\chi) + -X_1\cdot\frac{\alpha^2\cdot\omega_D}{\omega_d} + X_1\cdot\frac{\alpha^3}{\omega_d} + -X_1\cdot\frac{\alpha^2\cdot\omega_c(\chi)}{\omega_d} + X_1\cdot\frac{\alpha\cdot\omega_c(\chi)\cdot\omega_D}{\omega_d}\right)$$ **EQ 25**

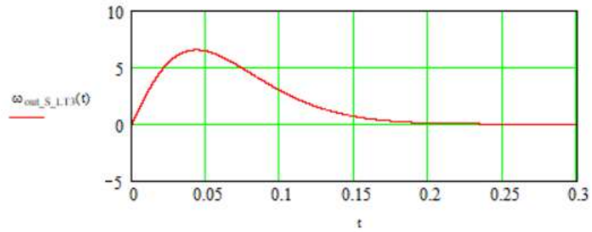
$$I_2(t) := \left(\frac{T_{Lp}}{J}\cdot\frac{kg\cdot m}{N}\cdot\frac{1}{D_{en}\cdot s^2}\right)\cdot\left[(K_{2A}(t))\cdot\cos(\omega_d\cdot t) + K_{2B}(t)\cdot\sin(\omega_d\cdot t)\right]$$ **EQ 26**

$$K_{3A}(t) := \exp(-\alpha\cdot t)\cdot X_2\cdot\alpha\cdot\omega_d$$ **EQ 27**

$$K_{3B}(t) := \exp(-\alpha\cdot t)\cdot X_1\cdot\alpha\cdot\omega_d$$ **EQ 28**

$$I_3(t) := \left(\frac{T_{Lp}}{J}\cdot\frac{kg\cdot m}{N}\cdot\frac{1}{D_{en}\cdot s^2}\right)\cdot\left(K_{3A}(t)\cdot\cos(\omega_d\cdot t) + K_{3B}(t)\cdot\sin(\omega_d\cdot t) + -\omega_c(\chi)\cdot\exp(-\omega_c(\chi)\cdot t) + \omega_D\cdot\exp(-\omega_c(\chi)\cdot t)\right)$$ **EQ 29**

$$\omega_{out\_S\_LT3}(t) := l_1(t) + l_2(t) + l_3(t)$$ **EQ 30**



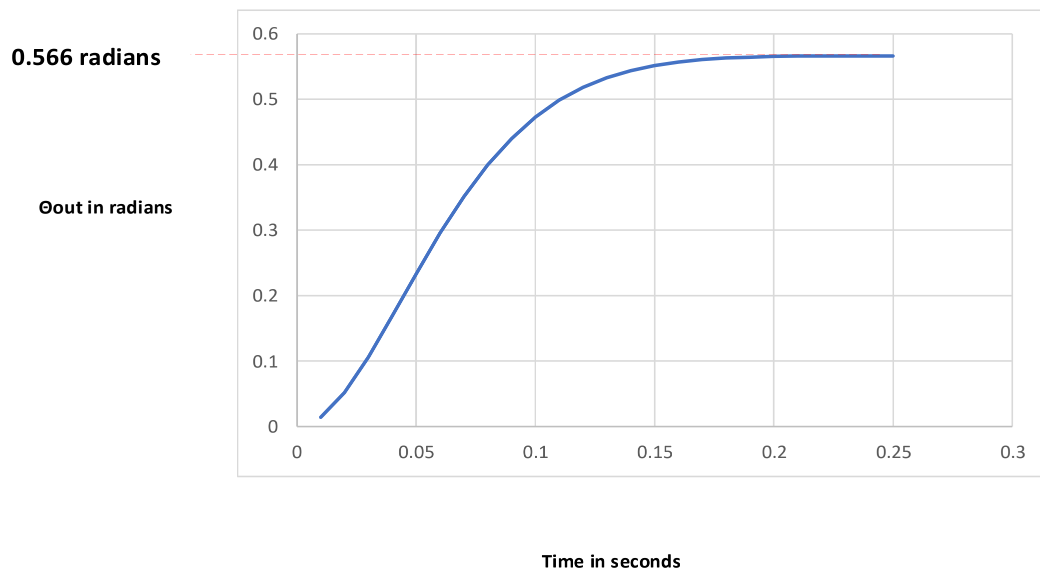
$$t_v := 0.02 \cdot s$$

$$T_v := \text{Maximize}(\omega_{out\_S\_LT3}, t_v)$$

Estimated		**Actual**
$T_{peak}(\chi, \omega_D) = 0.0435\,s$		$T_v = 0.0432\,s$
$\Delta\omega_o(\chi, \omega_D) = 6.511\,\frac{rad}{s}$	**ZERO if not in range** $\chi = 2.716$	$\omega_{out\_S\_LT3}(T_v) = 6.511\,\frac{rad}{s}$
$\dfrac{12.5}{\frac{\omega_D}{\chi} \cdot 1.68} = 0.202\,s$	**Recovery to within ~ 1% of peak**	$\omega_{out\_S\_LT3}(0.2005 \cdot s) = 0.065\,\frac{rad}{s}$

Note that at the end of the move due to the torque step disturbance a residual phase offset will result.

Plot for the above example is shown below.



This is the result of integrating the angular frequency plot (EQ 30 above, χ = 2.716).

$$\lim_{t_1 \to \infty} \int_0^{t_1} \omega_{out\_S\_LT3}(t)\,dt = 0.566 \cdot rad$$

Zone 3 **$3 < \chi < 25$** $\omega := 0\cdot\frac{rad}{s}, 0.1\cdot\frac{rad}{s} .. 800\cdot\frac{rad}{s}$

$3 < \chi < 25$ is split into two sub ranges;
$3 < \chi < 4$ and $4 <= \chi < 25$

+

Example $\chi := 4.0$

Identify sub ranges with the following Tests -

$A := 1$ $B := 0$

$IS\_CHI\_OK := [if[(3 < \chi < 25), A, B]]$

$IS\_CHI\_OK = 1$

P identifies $3 < \chi < 4$

$P := [if[(\chi > 3)\cdot IS\_CHI\_OK \wedge (\chi < 4)\cdot IS\_CHI\_OK, 1, 0]]$

$P = 0$ **P = 1 if in range, 0 otherwise**

Q identifies $4 <= \chi < 24$

$Q := [if[(\chi \geq 4)\cdot IS\_CHI\_OK \wedge (\chi < 25)\cdot IS\_CHI\_OK, 1, 0]]$

$Q = 1$ **Q = 1 if in range, 0 otherwise**

$R := P \wedge \neg Q$

$R = 0$

See **Page 10**, $\chi > 3$, $\zeta > 1$ in the document: *Cascade PI Controller Design using a Limited Span Method*.

$$\sigma(\chi) := \frac{\omega_D}{2\chi}(\chi - 1) \qquad \sigma(\chi) = 37.5\,\frac{\text{rad}}{\text{s}}$$

$$\omega_1(\chi) := \frac{\omega_D}{2\cdot\chi}\cdot\left(-1 + \chi - \sqrt{\chi^2 - 2\cdot\chi - 3}\right) \qquad \omega_1(\chi) = 9.549\,\frac{\text{rad}}{\text{s}}$$

$$\omega_2(\chi) := \frac{\omega_D}{2\cdot\chi}\cdot\left(-1 + \chi + \sqrt{\chi^2 - 2\cdot\chi - 3}\right) \qquad \omega_2(\chi) = 65.451\,\frac{\text{rad}}{\text{s}}$$

$$\omega_c(\chi) := \frac{\omega_D}{\chi} \qquad \omega_c(\chi) = 25\,\frac{\text{rad}}{\text{s}}$$

$$K_A(\chi) := \frac{\omega_D^{\,2}}{\chi^3} \qquad \omega_z(\chi) := \frac{\omega_D}{\chi^2}$$

$$K_A(\chi) = 156.25\,\frac{\text{rad}^2}{\text{s}^2} \qquad \omega_z(\chi) = 6.25\,\frac{\text{rad}}{\text{s}}$$

$$\omega_n := \omega_c(\chi) \qquad = \sqrt{\omega_1(\chi)\cdot\omega_2(\chi)} = 25\,\frac{\text{rad}}{\text{s}} \qquad \zeta := \frac{\sigma(\chi)}{\omega_n} \qquad \zeta = 1.5$$

See **Pages 21 and 22**, EQ 33 through EQ 36 in the document: *Cascade PI Controller Design using a Limited Span*

$$a := -0.35887268 \qquad \omega_{ref\_L} := 0.74967119\cdot\frac{\text{rad}}{\text{s}}$$

$$b := 0.46092839 \qquad \omega_{ref\_U} := 0.91066589\cdot\frac{\text{rad}}{\text{s}}$$

$$T_{peak}(\chi, \omega_D) := \left[\text{if}\left[(R = 1), \frac{\left(0.2163\cdot\chi^2 + 0.7558\cdot\chi + 0.6984\right)}{\omega_D}, \frac{\left(0.0409\cdot\chi^2 + 2.4526\cdot\chi - 3.685\right)}{\omega_D}\right]\right]\cdot(P \oplus Q)$$

$$\Delta\omega_o(\chi, \omega_D) := \left[\text{if}\left[(R = 1), (\chi - a)\cdot\left(\frac{1\cdot\frac{\text{rad}}{\text{s}}}{\omega_D}\cdot\frac{T_{Lp}}{1\cdot\text{N}\cdot\text{m}}\cdot\frac{1\cdot\text{kg}\cdot\text{m}^2}{\text{J}}\right)\cdot\omega_{ref\_L}, (\chi - b)\cdot\left(\frac{1\cdot\frac{\text{rad}}{\text{s}}}{\omega_D}\cdot\frac{T_{Lp}}{1\cdot\text{N}\cdot\text{m}}\cdot\frac{1\cdot\text{kg}\cdot\text{m}^2}{\text{J}}\right)\cdot\omega_{ref\_U}\right]\right]\cdot(P \oplus Q)$$

Time Domain response to an Impulse of torque

$3 < \chi < 25$ **For [$\zeta > 1$]**

$$\frac{(S+\omega_D)\cdot S}{(S+\omega_1(\chi))\cdot(S+\omega_c(\chi))\cdot(S+\omega_2(\chi))}\cdot\frac{T_{Lp}}{J}$$ **EQ 31** **Impulse Response**

$t := 0\cdot s, 0.01\cdot s .. 1.0\cdot s$ Time Range Variable

Let

$$\omega_{C1}(\chi) := \omega_c(\chi) - \omega_1(\chi)$$ **EQ 32**

$$\omega_{C2}(\chi) := \omega_c(\chi) - \omega_2(\chi)$$ **EQ 33**

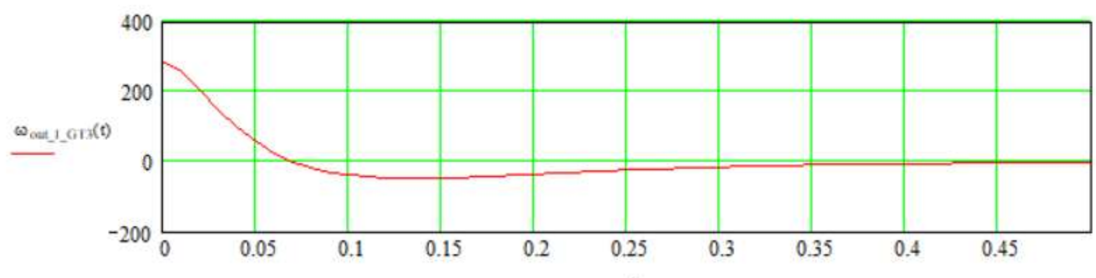
$$\omega_{12}(\chi) := \omega_1(\chi) - \omega_2(\chi)$$ **EQ 34**

$$g_1(t) := \left[\left(\frac{T_{Lp}\cdot kg\cdot m^2}{J\cdot N\cdot m}\cdot 1\cdot\frac{rad}{s}\right)\left[\frac{(\omega_1(\chi)\cdot\omega_D\cdot\exp(-\omega_1(\chi)\cdot t))}{\omega_{C1}(\chi)\cdot\omega_{12}(\chi)} - \frac{\omega_1(\chi)^2\cdot\exp(-\omega_1(\chi)\cdot t)}{\omega_{C1}(\chi)\cdot\omega_{12}(\chi)} - \frac{\omega_c(\chi)\cdot\omega_D\cdot\exp(-\omega_c(\chi)\cdot t)}{\omega_{C2}(\chi)\cdot\omega_{C1}(\chi)}\right]\right]$$ **EQ 35**

$$g_2(t) := \left[\left(\frac{T_{Lp}\cdot kg\cdot m^2}{J\cdot N\cdot m}\cdot 1\cdot\frac{rad}{s}\right)\left(\frac{\omega_c(\chi)^2\cdot\exp(-\omega_c(\chi)\cdot t)}{\omega_{C2}(\chi)\cdot\omega_{C1}(\chi)} - \frac{\omega_2(\chi)\cdot\omega_D\cdot\exp(-\omega_2(\chi)\cdot t)}{\omega_{C2}(\chi)\cdot\omega_{12}(\chi)} + \frac{\omega_2(\chi)^2\cdot\exp(-\omega_2(\chi)\cdot t)}{\omega_{C2}(\chi)\cdot\omega_{12}(\chi)}\right)\right]$$ **EQ 36**

+

$$\omega_{out\_I\_GT3}(t) := g_1(t) + g_2(t)$$ **EQ 37**



$\omega_{out\_I\_GT3}(0\cdot s) = 282.446\,\frac{rad}{s}$ peak velocity

Time Domain Response to an Step of Torque

$3 < \chi <$ **25** **For [ζ > 1]**

$$\frac{(S + \omega_D)}{(S + \omega_1(\chi))\cdot(S + \omega_c(\chi))\cdot(S + \omega_2(\chi))}\cdot\frac{T_{Lp}}{J}$$ **EQ 38** **Step Response**

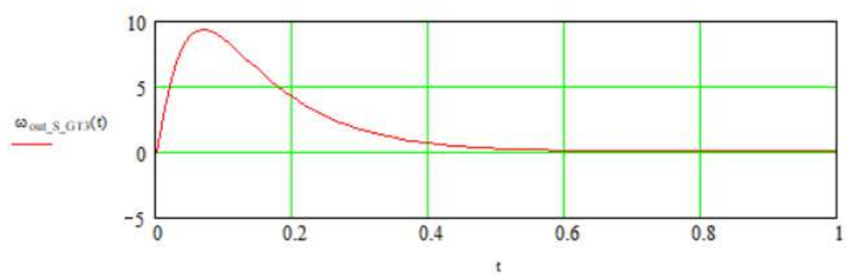
Let

$$h_1(t) := \left(\frac{T_{Lp}}{J}\cdot\frac{kg\cdot m^2}{N\cdot m}\cdot\frac{rad}{s^2}\right)\cdot\frac{(\omega_1(\chi) - \omega_D)}{\omega_{C1}(\chi)\cdot\omega_{12}(\chi)}\cdot\exp(-\omega_1(\chi)\cdot t)$$ **EQ 39**

$$h_2(t) := \left(\frac{T_{Lp}}{J}\cdot\frac{kg\cdot m^2}{N\cdot m}\cdot\frac{rad}{s^2}\right)\cdot\frac{(\omega_D - \omega_c(\chi))}{\omega_{C1}(\chi)\cdot\omega_{C2}(\chi)}\cdot\exp(-\omega_c(\chi)\cdot t)$$ **EQ 40**

$$h_3(t) := \left(\frac{T_{Lp}}{J}\cdot\frac{kg\cdot m^2}{N\cdot m}\cdot\frac{rad}{s^2}\right)\cdot\frac{(\omega_D - \omega_2(\chi))}{\omega_{C2}(\chi)\cdot\omega_{12}(\chi)}\cdot\exp(-\omega_2(\chi)\cdot t)$$ **EQ 41**

$$\omega_{out\_S\_GT3}(t) := h_1(t) + h_2(t) + h_3(t)$$ **EQ 42**



$T_v := \text{Maximize}(\omega_{out\_S\_GT3}, t_v)$

Estimated **Actual**

$T_{peak}(\chi, \omega_D)\cdot(P \oplus Q) = 0.0678\,s$ $T_v = 0.06981\,s$

Both are zero if out of range.

$\Delta\omega_o(\chi, \omega_D) = 9.103\,\frac{rad}{s}$ $\omega_{out\_S\_GT3}(T_v) = 9.313\,\frac{rad}{s}$

$\frac{5.2}{\omega_1(\chi)} = 0.5446\,s$ **Recovery to within ~ 1% of peak** $\omega_{out\_S\_GT3}(0.60331\cdot s) = 0.09309\,\frac{rad}{s}$

$\frac{0.5446 - 0.60331}{0.60331} = -9.731\,\%$ **Error in Recovery Estimate**

This is the result of integrating the angular frequency plot (EQ 42 above, χ = 4.0).

$$\lim_{t_1 \to \infty}\int_0^{t_1}\omega_{out\_S\_GT3}(t)\,dt = 1.808\cdot rad$$

Sinusoidal Response

Frequency Domain

$$\frac{T_\xi(S)}{T_L(S)} = \frac{S^2\cdot(S+\omega_D)}{S^3 + S^2\cdot\omega_D + S\cdot\frac{\omega_D^{\,2}}{\chi} + \frac{\omega_D^{\,3}}{\chi^3}} \qquad \textbf{EQ 43}$$

$$\frac{\omega_o(S)}{T_L(S)} = \frac{1}{J}\cdot\frac{S\cdot(S+\omega_D)}{S^3 + S^2\cdot\omega_D + S\cdot\frac{\omega_D^{\,2}}{\chi} + \frac{\omega_D^{\,3}}{\chi^3}} \qquad \textbf{EQ 44}$$

$S = j\cdot\omega$

$$G_\xi(\omega) = \frac{T_\xi(\omega)}{T_L(\omega)} \qquad \textbf{EQ 45}$$

+

$$G_\xi(\omega) := \frac{(j\cdot\omega)^2\cdot\left(1 + j\cdot\frac{\omega}{\omega_D}\right)\cdot\omega_D}{\left(1 + j\cdot\frac{\omega}{\omega_1(\chi)}\right)\cdot\left(1 + j\cdot\frac{\omega}{\omega_c(\chi)}\right)\cdot\left(1 + j\cdot\frac{\omega}{\omega_2(\chi)}\right)\cdot\omega_1(\chi)\cdot\omega_c(\chi)\cdot\omega_2(\chi)} \qquad \textbf{EQ 46}$$

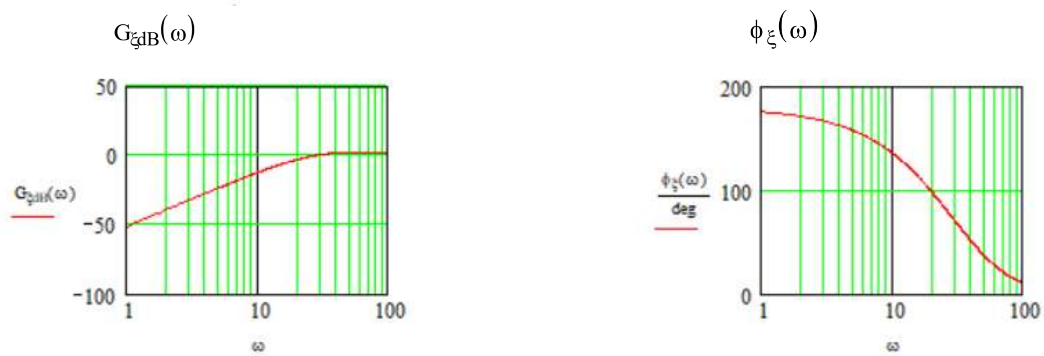
$$G_{\xi dB}(\omega) := 20\cdot\log\left(\left|G_\xi(\omega)\right|\right) \qquad \textbf{EQ 47}$$

$$\phi_\xi(\omega) := \arg\left(G_\xi(\omega)\right) \qquad \textbf{EQ 48}$$

Zone 1

$\chi = 3$ $\qquad \omega_D = 100\,\frac{rad}{s}$

$\omega_1(\chi) = 33.333\,\frac{rad}{s}$ $\qquad \omega_c(\chi) = 33.333\,\frac{rad}{s}$ $\qquad \omega_2(\chi) = 33.333\,\frac{rad}{s}$



A Sinusoid of Torque

$T_L(t) := T_{Lp} \cdot \cos(\omega_{test} \cdot t)$

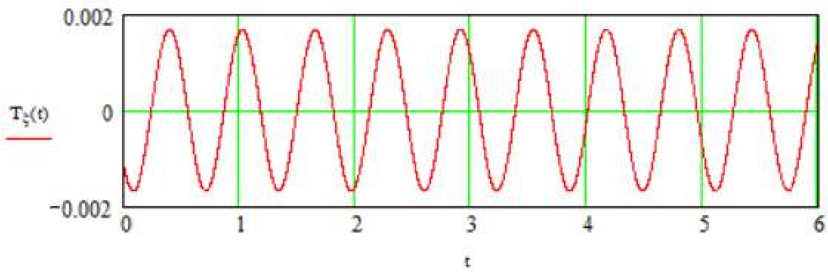
Where $T_{Lp} = 7.061 \times 10^{-3}\, N \cdot m$ $\omega_{test} = 10 \frac{rad}{s}$

$t := 0 \cdot s, 0.001 \cdot s .. 6 \cdot s$

$T_{Px} := T_{Lp} \cdot |G_\xi(\omega_{test})|$ $\Theta := \arg(G_\xi(\omega_{test}))$

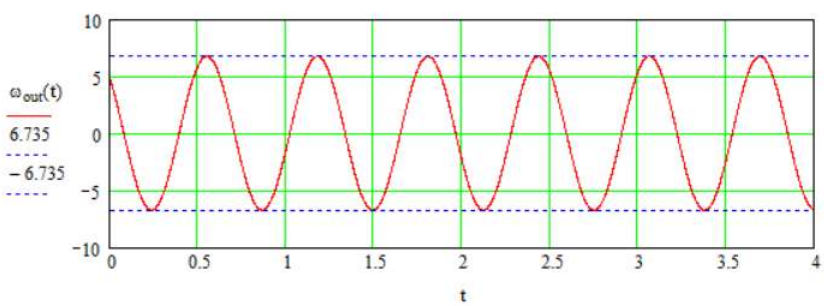
$T_{Px} = 1.684 \times 10^{-3}\, N \cdot m$ $\Theta = 2.367\, rad$ $\Theta = 135.613\, deg$

$T_\xi(t) := T_{Px} \cdot \cos(\omega_{test} \cdot t + \Theta)$



Therefore

$$\omega_{out}(t) := \frac{1}{J} \cdot \frac{T_{Px}}{\omega_{test}} \cdot \sin(\omega_{test} \cdot t + \Theta)$$



Peak of the output

$Ampl := \frac{1}{J} \cdot \frac{T_{Px}}{\omega_{test}}$

$Ampl = 6.735 \frac{rad}{s}$

Magnitude

$\omega_{out}(1.805 \cdot s) = 6.735 \frac{rad}{s}$

$\omega_{out}(2.433 \cdot s) = 6.735 \frac{rad}{s}$

$\omega_{out}(2.119 \cdot s) = -6.735 \frac{rad}{s}$

$\frac{2 \cdot \pi}{2.433 \cdot s - 1.805 \cdot s} = 10.005 \frac{rad}{s}$

Period $T := \frac{1}{1.59155 \cdot Hz}$ $T = 0.628\, s$

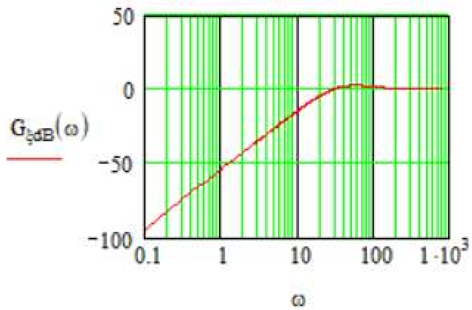
Zone 2

$$\chi = 2.716 \qquad \omega_D = 100\,\frac{\text{rad}}{\text{s}}$$

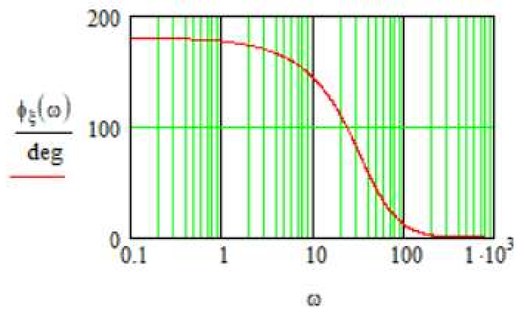
$$\omega_1(\chi) = 31.591 - 18.912i\,\frac{\text{rad}}{\text{s}} \qquad \omega_c(\chi) = 36.819\,\frac{\text{rad}}{\text{s}} \qquad \omega_2(\chi) = 31.591 + 18.912i\,\frac{\text{rad}}{\text{s}}$$

Using equations EQ 10 through EQ 15 results in the following amplitude/phase plots below:

$G_{\xi dB}(\omega)$

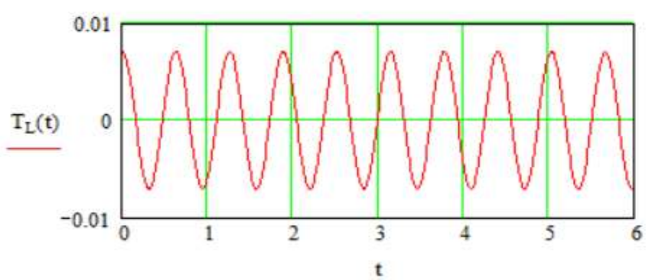


$\phi_\xi(\omega)$



Input Torque Disturbance: $t := 0\cdot s, 0.01\cdot s .. 6\cdot s$ Time Range Variable

$$T_L(t) := T_{Lp}\cdot\cos(\omega_{test}\cdot t)$$

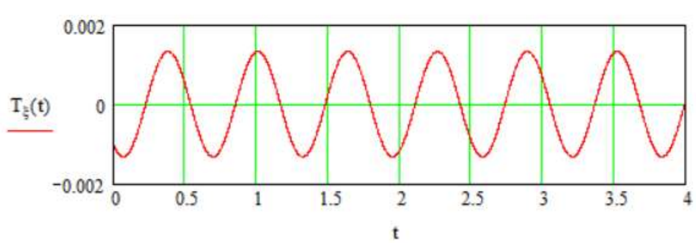


Torque Summing Point

$$T_{Px} := T_{Lp}\cdot\left|G_\xi(\omega_{test})\right| \qquad \Theta := \arg(G_\xi(\omega_{test}))$$

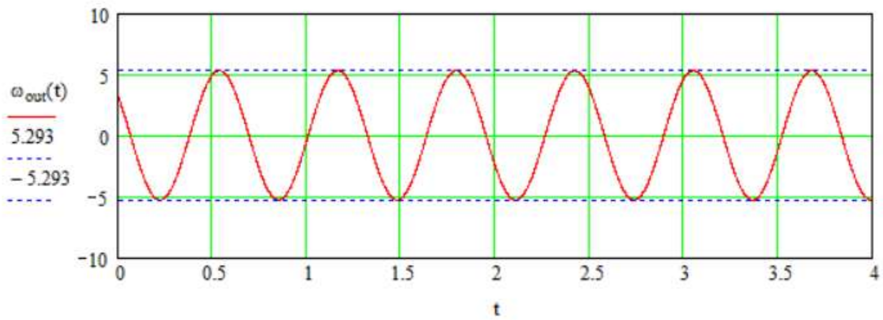
$$T_{Px} = 1.323\times10^{-3}\,\text{N}\cdot\text{m} \qquad \Theta = 2.51\,\text{rad} \qquad \Theta = 143.805\,\text{deg}$$

$$T_\xi(t) := T_{Px}\cdot\cos(\omega_{test}\cdot t + \Theta)$$



Therefore

$$\omega_{out}(t) := \frac{1}{J} \cdot \frac{T_{Px}}{\omega_{test}} \cdot \sin\left(\omega_{test} \cdot t + \Theta\right)$$



Peak of the output

$$\text{Ampl} := \frac{1}{J} \cdot \frac{T_{Px}}{\omega_{test}}$$

$$\text{Ampl} = 5.293 \frac{\text{rad}}{\text{s}}$$

Magnitude

$$\omega_{out}(2.42 \cdot \text{s}) = 5.293 \frac{\text{rad}}{\text{s}} \qquad \frac{2 \cdot \pi}{3.048 \cdot \text{s} - 2.42 \cdot \text{s}} = 10.005 \frac{\text{rad}}{\text{s}}$$

$$\omega_{out}(3.048 \cdot \text{s}) = 5.293 \frac{\text{rad}}{\text{s}}$$

$$\omega_{out}(2.734 \cdot \text{s}) = -5.293 \,\text{Hz}$$

period $T := \dfrac{2 \cdot \pi}{10 \cdot \dfrac{\text{rad}}{\text{s}}}$ $\qquad T = 0.628 \,\text{s}$

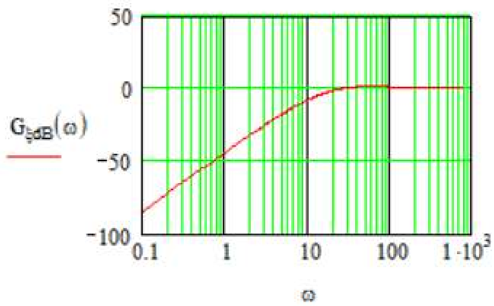
Zone 3

$\chi = 4$ $\qquad \omega_D = 100\,\frac{rad}{s}$ $\qquad \omega_{test} = 10\,\frac{rad}{s}$

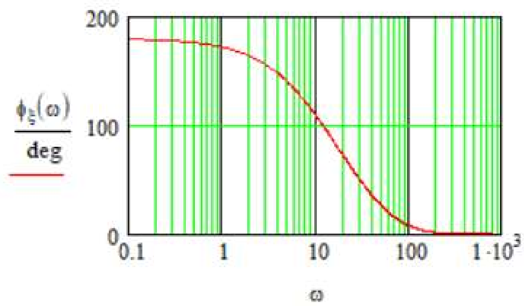
$\omega_1(\chi) = 9.549\,\frac{rad}{s}$ $\qquad \omega_c(\chi) = 25\,\frac{rad}{s}$ $\qquad \omega_2(\chi) = 65.451\,\frac{rad}{s}$

Using equations EQ 10 through EQ 15 results in the following amplitude/phase plots below:

$G_{\xi dB}(\omega)$



$\phi_\xi(\omega)$



Torque Summing Point

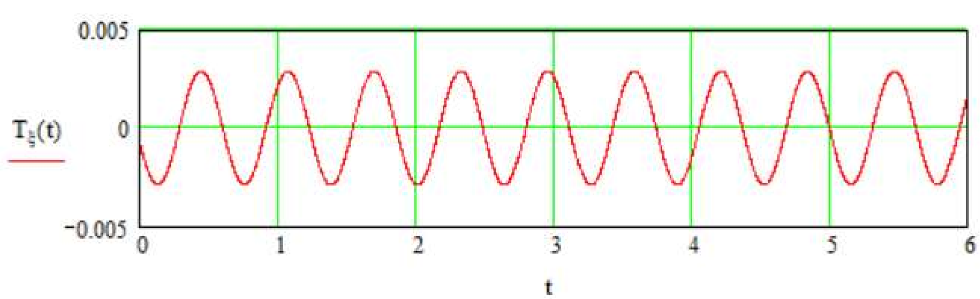
$T_L(t) := T_{Lp} \cdot \cos(\omega_{test} \cdot t)$ $\qquad T_{Lp} = 7.061 \times 10^{-3}\,N \cdot m$

$t := 0 \cdot s, 0.001 \cdot s .. 6 \cdot s$

$T_{Px} := T_{Lp} \cdot \left|G_\xi(\omega_{test})\right|$ $\qquad \Theta := \arg(G_\xi(\omega_{test}))$

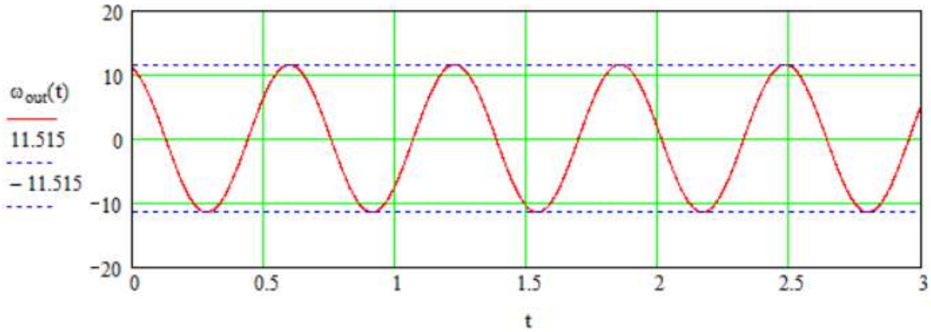
$T_{Px} = 2.879 \times 10^{-3}\,N \cdot m$ $\qquad \Theta = 1.901\,rad$ $\qquad \Theta = 108.901\,deg$

$T_\xi(t) := T_{Px} \cdot \cos(\omega_{test} \cdot t + \Theta)$



Therefore

$$\omega_{out}(t) := \frac{1}{J}\cdot\frac{T_{Px}}{\omega_{test}}\cdot\sin\left(\omega_{test}\cdot t+\Theta\right)$$



Peak of the output

$$\frac{T_{Px}}{J}\cdot\frac{1}{\omega_{test}} = 11.515\,\frac{rad}{s}$$

Magnitude

$$\omega_{out}(1.224\cdot s) = 11.515\,\frac{rad}{s}$$

$$\omega_{out}(1.852\cdot s) = 11.515\,\frac{rad}{s}$$

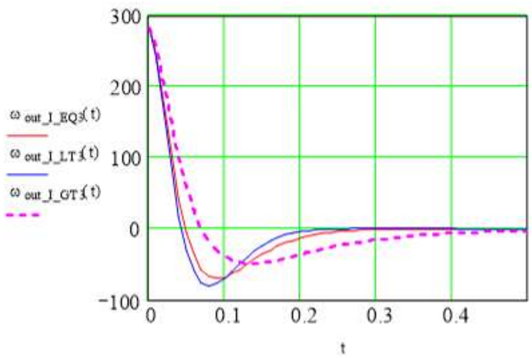
$$\omega_{out}(1.538\cdot s) = -11.515\,\frac{rad}{s}$$

$$\frac{2\cdot\pi}{1.852\cdot s-1.224\cdot s} = 10.005\,\frac{rad}{s}$$

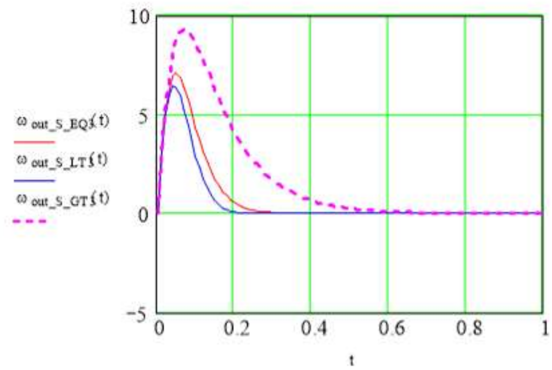
Period $$T := \frac{1}{1.59155\cdot Hz} \qquad T = 0.628\,s$$

Comparison Summary

Impulse Response Comparisons

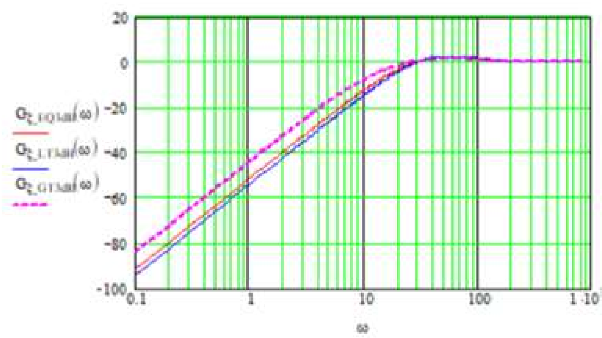


Step Response Comparisons

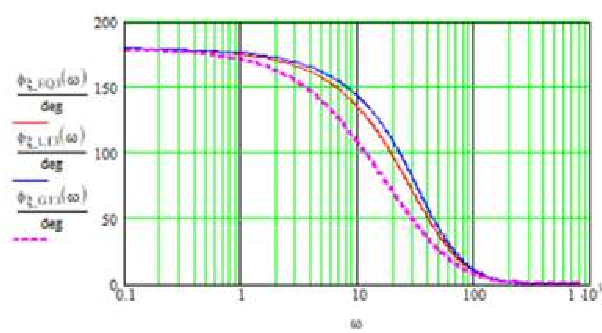


Frequence Domain Comparisons

Amplitude



Phase



Legend

$\chi := 3$	**EQ3**
$\chi := 2.716$	**LT3**
$\chi := 4$	**GT3**

Comparison Summary
(continued)

Considering the residual position from a step of torque:

Repeat the equations -

$$S_{\xi\_spd}(S) := \frac{S^2\cdot(S+\omega_D)}{S^3+S^2\cdot\omega_D+S\cdot\frac{\omega_D^{\ 2}}{\chi}+\frac{\omega_D^{\ 3}}{\chi^3}} \qquad \textbf{EQ 6}$$

$$\frac{\omega_{out}(S)}{T_L(S)} = \frac{1}{S\cdot J}\cdot S_{\xi\_spd}(S) \qquad \textbf{EQ 7}$$

Rearranging and Integrating EQ 7

$$\theta_{out\_spd}(S) = \frac{1}{S^2\cdot J}\cdot S_{\xi\_spd}(S)\cdot T_L(S)$$

Using the Final Value Theorem for Laplace Transforms which is:

$$\lim_{S\to 0} S\cdot Q(S) = \lim_{t\to\infty} q(t)$$

When TL(S) = T_{Lp} (Impulse of Torque)

$$\theta_{out\_spd\_I}(\infty) = \lim_{S\to 0}\frac{S\cdot(S+\omega_D)}{S^3+S^2\cdot\omega_D+S\cdot\frac{\omega_D}{\chi^2}+\frac{\omega_D^{\ 3}}{\chi^3}}\cdot\frac{T_{Lp}}{J} = 0 \qquad \textbf{EQ 49}$$

Then TL(S) = T_{Lp} / S (Step of Torque)

$$\theta_{out\_spd\_S}(\infty) = \lim_{S\to 0}\frac{(S+\omega_D)}{S^3+S^2\cdot\omega_D+S\cdot\frac{\omega_D}{\chi^2}+\frac{\omega_D^{\ 3}}{\chi^3}}\cdot\frac{T_{Lp}}{J} = \frac{\chi^3}{\omega_D^{\ 2}}\cdot\frac{T_{Lp}}{J} \qquad \textbf{EQ 50}$$

$$\theta_{out\_spd\_S\_final}(\chi) := \frac{\chi^3}{\omega_D^{\ 2}}\cdot\frac{T_{Lp}}{J} \qquad \textbf{EQ 51}$$

Matches the results on the referenced Pages

$\theta_{out\_spd\_final}(2.716) = 0.566\,\text{rad}$ See Page 12

$\theta_{out\_spd\_final}(3) = 0.763\,\text{rad}$ See Page 7

$\theta_{out\_spd\_final}(4) = 1.808\,\text{rad}$ See Page 16

However, note that when integrating the velocity Impulse response, the following is true:
(For the same Impulse and Step function amplitude / area value and Inertia)

X = LT3 or EQ3 or GT3

With $T_v = \text{Maximize}(\omega_{out\_S\_X}, t_v)$

$$\frac{\int_0^{T_v} \omega_{out\_I\_X}(t)\,dt}{1 \cdot s} = \omega_{out\_S\_X}(T_v) \qquad \textbf{EQ 52}$$

T_v is the crossover time (ωout = 0) for the Impulse Response as well as the peak time for the Step Response!

That is

$$\frac{\left(\int_{0 \cdot s}^{T_v} \omega_{out\_I\_EQ3}(t)\,dt\right)}{1 \cdot s} = 7.117\,\frac{\text{rad}}{\text{s}} \qquad T_v = 0.049\,\text{s}$$

$$\omega_{out\_S\_EQ3}(T_v) = 7.117\,\frac{\text{rad}}{\text{s}}$$

$$\frac{\left(\int_{0 \cdot s}^{T_v} \omega_{out\_I\_LT3}(t)\,dt\right)}{1 \cdot s} = 6.511\,\frac{\text{rad}}{\text{s}} \qquad T_v = 0.043\,\text{s}$$

$$\omega_{out\_S\_LT3}(T_v) = 6.511\,\frac{\text{rad}}{\text{s}}$$

$$\frac{\int_{0 \cdot s}^{T_v} \omega_{out\_I\_GT3}(t)\,dt}{1 \cdot s} = 9.313\,\frac{\text{rad}}{\text{s}} \qquad T_v = 0.0698\,\text{s}$$

$$\omega_{out\_S\_GT3}(T_v) = 9.313\,\frac{\text{rad}}{\text{s}}$$