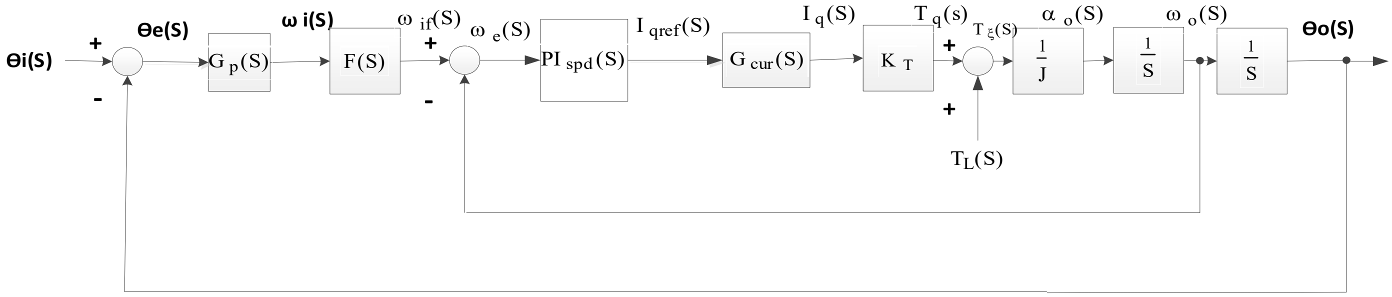
Cascade Position PI Loop Torque Disturbance Analysis:

$j := \sqrt{-1}$ $\qquad$ $dB := 1$



Torque Disturbance Model for a Cascade PI Position Loop

Figure 1a

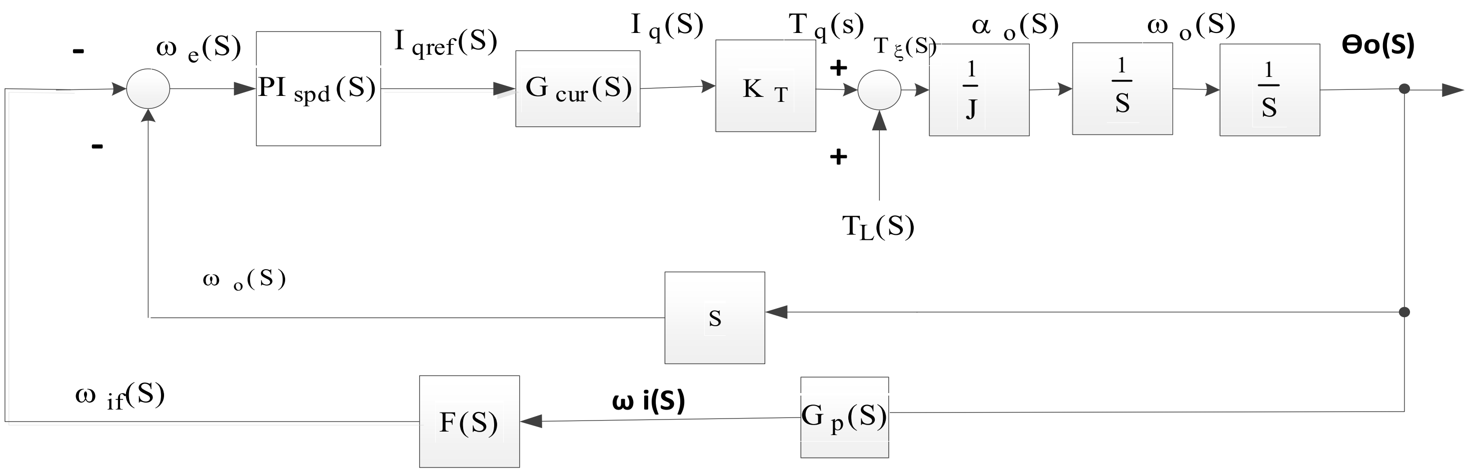


Figure 1b

Equivalent Model

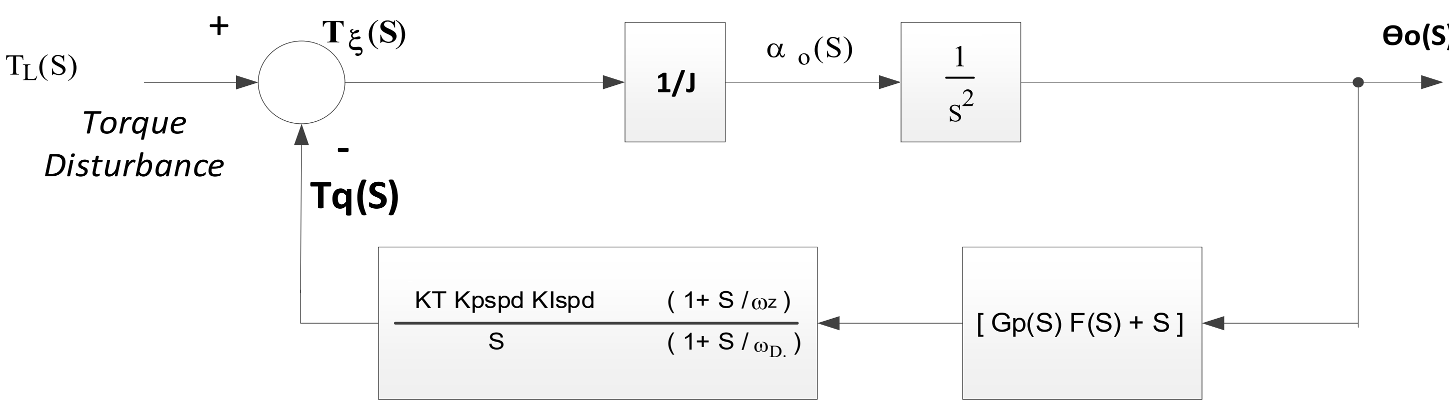


Figure 1c

Figures 1a, b and c Torque Disturbance Models for a Position Loop

$$G_P(S) = K_V$$

$$F(S) = \frac{\omega_z}{(S + \omega_z)}$$

$$PI_{spd}(S) = \frac{K_{pspd} \cdot K_{Ispd}}{S} \cdot \frac{(S + \omega_z)}{\omega_z}$$

$$G_{cur}(S) = \frac{\omega_D}{(S + \omega_D)}$$

$$T_\xi(S) = J \cdot \alpha(S) + D \cdot \omega(S)$$

J (kg m^2) is the total moment of inertia.

$$\alpha = \frac{d}{dt}\omega(t) \qquad \frac{\left(\frac{rad}{s}\right)}{s}$$

Angular Acceleration.

$$D \qquad \frac{N \cdot m}{\left(\frac{rad}{s}\right)}$$

Viscus Friction constant.

If we can neglect the viscus friction term -

$$\alpha(S) = \frac{T_\xi(S)}{J} \qquad \left[\frac{\left(\frac{rad}{s}\right)}{s}\right] \qquad \frac{N \cdot m}{kg \cdot m^2} = 1\,\frac{\left(\frac{rad}{s}\right)}{s}$$

$$G_{OL\_pos}(S) = \frac{K_{pspd} \cdot K_{Ispd}}{S} \cdot \frac{(S + \omega_z)}{\omega_z} \cdot \frac{\omega_D}{(S + \omega_D)} \cdot K_T \cdot \frac{1}{J} \cdot \frac{1}{S^2} \cdot \left[S + K_V \cdot \frac{\omega_z}{(S + \omega_z)} \right]$$

Since $K_A = \frac{K_T}{J} \cdot K_{pspd} \cdot K_{Ispd}$

$$G_{OL\_pos}(S) = K_A \cdot \frac{\omega_D}{S^3 \cdot (S + \omega_D)} \cdot \left[S \cdot \frac{(S + \omega_z)}{\omega_z} + K_V \right] = \frac{K_A \cdot \omega_D}{S^2 \cdot (S + \omega_D)} \cdot \left[\frac{K_V}{S} + \frac{(S + \omega_z)}{\omega_z} \right]$$

and

$$\frac{\omega_D}{\omega_z} = \chi^2 \qquad K_A \cdot \chi^2 = \frac{\omega_D^2}{\chi}$$

$$\frac{1}{1 + G_{OL\_pos}(S)} = \frac{S^3 \cdot (S + \omega_D)}{S^4 + S^3 \cdot \omega_D + S^2 \cdot \frac{\omega_D^2}{\chi} + S \cdot \frac{\omega_D^3}{\chi^3} + K_V \frac{\omega_D^3}{\chi^3}} = S_{\xi\_pos}(S)$$

$S_{\xi\_pos}(S)$ is the Position Loop Sensitivity Function

Disturbance Transfer Function is given by :

$$\frac{\theta_o(S)}{T_L(S)} = \frac{\frac{1}{J}\cdot\frac{1}{S^2}}{1 + G_{OL\_pos}(S)} = \frac{1}{J\cdot S^2}\cdot S_{\xi\_pos}(S)$$

$$\theta_o(S) = \frac{1}{J}\cdot\frac{1}{S^2}\cdot\frac{S^3\cdot(S+\omega_D)}{S^4 + S^3\cdot\omega_D + S^2\cdot\frac{\omega_D^2}{\chi} + S\cdot\frac{\omega_D^3}{\chi^3} + K_V\cdot\frac{\omega_D^3}{\chi^3}}\cdot T_L(S)$$

$$\theta_o(S) = \frac{1}{J}\cdot\frac{S\cdot(S+\omega_D)}{S^4 + S^3\cdot\omega_D + S^2\cdot\frac{\omega_D^2}{\chi} + S\cdot\frac{\omega_D^3}{\chi^3} + K_V\cdot\frac{\omega_D^3}{\chi^3}}\cdot T_L(S)$$

Conversion Factors

$$ounce\_in := \frac{1\cdot N\cdot m}{141.61193} \qquad Rev := 2\cdot\pi\cdot rad \qquad RPM := 1\cdot\frac{Rev}{60\cdot s}$$

This Example uses the χ = 2.716 parameters used in the document - *Cascade PI Velocity Controler Torque Disturbance Analysis* by Frederick York. That document treats Velocity Loop performance while herein is a Position Loop response.

Defined Inputs

$$\omega := 0\cdot\frac{rad}{s}, 0.01\cdot\frac{rad}{s} .. 200\cdot\frac{rad}{s} \qquad t := 0\cdot s, 0.0001\cdot s .. 3\cdot s$$

$J := 2.5\cdot 10^{-5}\cdot kg\cdot m^2$ Total Moment of Inertia $\qquad \omega_D := 100\cdot\frac{rad}{s}$

$T_{Lp} := 7.0611519\cdot 10^{-3}\cdot N\cdot m$ Amplitude of the Torque Disturbance

$T_{Lp} = 1\ ounce\_in$

From the Model

$T_\xi(S) = T_L(S) + T_q(S)$ **Summing Point for the torque disturbance**

$$\frac{\theta_o(S)}{T_l(S)} = \frac{1}{J}\cdot\frac{S\cdot(S+\omega_D)}{S^4 + S^3\cdot\omega_D + S^2\cdot\frac{\omega_D^2}{\chi} + S\cdot\frac{\omega_D^3}{\chi^3} + K_V\cdot\frac{\omega_D^3}{\chi^3}}$$

$\omega_D = 100 \frac{rad}{s}$ $\quad K_V := 1.885 \cdot \frac{1}{s}$ $\quad \chi := 2.716$

$\omega_z := \frac{\omega_D}{\chi^2}$ $\quad \omega_z = 13.556 \frac{rad}{s}$ $\quad K_x := \frac{T_{Lp}}{J}$

$Q(S) := S^4 + 100 \cdot S^3 + 3681.885125 \cdot S^2 + 49912.6586 \cdot S + 94085.36146$

Characteristic Eqution

$$Q(S) \text{ solve}, S \rightarrow \begin{pmatrix} -35.030366027170446073 - 17.187083375305704500 \cdot i \\ -35.030366027170446073 + 17.187083375305704500 \cdot i \\ -27.709109027622025737 \\ -2.2301589180370821171 \end{pmatrix}$$

$\sigma_1 := 2.230158918 \cdot \frac{rad}{s}$ $\quad \sigma_3 := 35.03036602 \cdot \frac{rad}{s}$

Roots (Poles)

$\sigma_2 := 27.7091093 \cdot \frac{rad}{s}$ $\quad \omega_3 := 17.1870834 \cdot \frac{rad}{s}$

$$\frac{(S + \omega_D) \cdot K_x}{(S + \sigma_1) \cdot (S + \sigma_2) \cdot (S + \sigma_3 - j \cdot \omega_3) \cdot (S + \sigma_3 + j \cdot \omega_3)} = \frac{P(S)}{Q(S)}$$

Step Response Transfer Function

$$\frac{(S + \omega_D) \cdot K_x}{(S + \sigma_1) \cdot (S + \sigma_2) \cdot \left[(S + \alpha)^2 + \omega_d^{\,2}\right]}$$

$$\frac{k_1}{(S + \sigma_1)} + \frac{k_2}{(S + \sigma_2)} + \frac{k_3}{(S + \sigma_3 + j \cdot \omega_3)} + \frac{\overline{k_3}}{(S + \sigma_3 - j \cdot \omega_3)}$$

Solve Partial Fraction Expansion to determine the Time Domain Response.

$\omega_n := \sqrt{\sigma_3^{\,2} + \omega_3^{\,2}}$ $\quad \zeta := \frac{\sigma_3}{\omega_n}$ $\quad \sigma_z := \frac{\omega_D}{\chi^2}$

$\alpha := \zeta \cdot \omega_n$ $\quad \omega_d := \omega_n \cdot \sqrt{1 - \zeta^2}$

$\omega_n = 39.02 \frac{rad}{s}$ $\quad \zeta = 0.898$ $\quad \frac{\sigma_3}{\sigma_1} = 15.708$ $\quad \frac{\sigma_2}{\sigma_1} = 12.425$

$\alpha = 35.03 \frac{rad}{s}$ $\quad \omega_d = 17.187 \frac{rad}{s}$

Time Domain Analysis

$$k_1 = \frac{(S+\sigma_1)\cdot(S+\omega_D)\cdot K_x}{D(S)} \Bigg|_{S=-\sigma_1}$$

$$D(S) := (S+\sigma_1)\cdot(S+\sigma_2)\cdot(S+\sigma_3+j\cdot\omega_3)\cdot(S+\sigma_3-j\cdot\omega_3)$$

$$D_{k1}(S) = (S+\sigma_2)\cdot(S+\sigma_3+j\cdot\omega_3)\cdot(S+\sigma_3-j\cdot\omega_3)$$

$$D_{k1} := (-\sigma_1+\sigma_2)\cdot(-\sigma_1+\sigma_3+j\cdot\omega_3)\cdot(-\sigma_1+\sigma_3-j\cdot\omega_3) \qquad S = -\sigma_1 \qquad -\sigma_1 = -2.23\,\frac{rad}{s}$$

$$D_{k1} = 3.494\times 10^4\,\frac{1}{s^3}$$

$$\mathrm{Re}\left(-\sigma_1^{\,3} + 2\cdot\sigma_1^{\,2}\cdot\sigma_3 - \sigma_1\cdot\sigma_3^{\,2} + \sigma_1\cdot j^2\cdot\omega_3^{\,2} + \sigma_2\cdot\sigma_1^{\,2} - 2\cdot\sigma_2\cdot\sigma_1\cdot\sigma_3 + \sigma_2\cdot\sigma_3^{\,2} - \sigma_2\cdot j^2\cdot\omega_3^{\,2}\right) = 3.494\times 10^4\,\frac{1}{s^3}$$

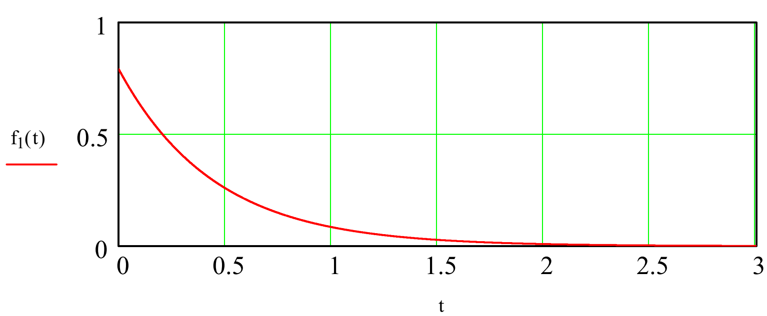
$$\mathrm{Im}\left(\left(-\sigma_1^{\,3} + 2\cdot\sigma_1^{\,2}\cdot\sigma_3 - \sigma_1\cdot\sigma_3^{\,2} + \sigma_1\cdot j^2\cdot\omega_3^{\,2} + \sigma_2\cdot\sigma_1^{\,2} - 2\cdot\sigma_2\cdot\sigma_1\cdot\sigma_3 + \sigma_2\cdot\sigma_3^{\,2} - \sigma_2\cdot j^2\cdot\omega_3^{\,2}\right)\right) = 0\,\frac{1}{s^3}$$

$P_{k1}(S)$

$$P_{k1} := (-\sigma_1+\omega_D)\cdot K_x \qquad P_{k1} = 2.761\times 10^4\,\frac{1}{s^3}$$

$$k_1 := \frac{P_{k1}}{D_{k1}} \qquad k_1 = 0.79$$

$$f_1(t) := k_1\cdot\exp(-\sigma_1\cdot t)$$



$$k_2 = \left. \frac{(S+\sigma_2)\cdot(S+\omega_D)\cdot K_x}{D_{k2}(S)} \right|_{S = -\sigma_2}$$

$$\frac{(S+\sigma_2)}{D(S)} = (S+\sigma_1)\cdot(S+\sigma_3+j\cdot\omega_3)\cdot(S+\sigma_3-j\cdot\omega_3) \qquad -\sigma_2 = -27.709\,\frac{rad}{s}$$

$$D(S) := (S+\sigma_1)\cdot(S+\sigma_2)\cdot(S+\sigma_3+j\cdot\omega_3)\cdot(S+\sigma_3-j\cdot\omega_3)$$

$$D_{k2}(S) := (S+\sigma_1)\cdot(S+\sigma_3+j\cdot\omega_3)\cdot(S+\sigma_3-j\cdot\omega_3)$$

$$D_{k2} := (-\sigma_2+\sigma_1)\cdot(-\sigma_2+\sigma_3+j\cdot\omega_3)\cdot(-\sigma_2+\sigma_3-j\cdot\omega_3)$$

$S = j\cdot\omega$

$$D_{k2} = -8.892\times 10^3\,\frac{1}{s^3}$$

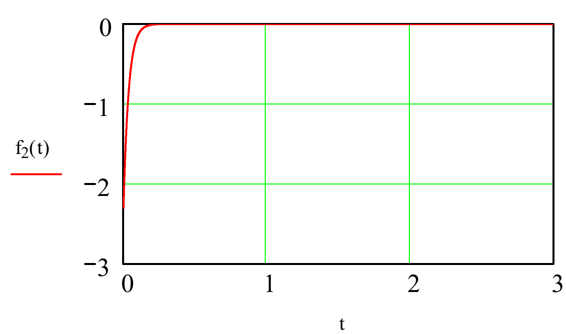
$$\mathrm{Re}\left(-\sigma_2^{\,3} + 2\cdot\sigma_2^{\,2}\cdot\sigma_3 - \sigma_2\cdot\sigma_3^{\,2} + \sigma_2\cdot j^2\cdot\omega_3^{\,2} + \sigma_1\cdot\sigma_2^{\,2} - 2\cdot\sigma_1\cdot\sigma_2\cdot\sigma_3 + \sigma_1\cdot\sigma_3^{\,2} - \sigma_1\cdot j^2\cdot\omega_3^{\,2}\right) = -8.892\times 10^3\,\frac{1}{s^3}$$

$$\mathrm{Im}\left(-\sigma_2^{\,3} + 2\cdot\sigma_2^{\,2}\cdot\sigma_3 - \sigma_2\cdot\sigma_3^{\,2} + \sigma_2\cdot j^2\cdot\omega_3^{\,2} + \sigma_1\cdot\sigma_2^{\,2} - 2\cdot\sigma_1\cdot\sigma_2\cdot\sigma_3 + \sigma_1\cdot\sigma_3^{\,2} - \sigma_1\cdot j^2\cdot\omega_3^{\,2}\right) = 0\,\frac{1}{s^3}$$

$$P_{k2} := (-\sigma_2+\omega_D)\cdot K_x \qquad P_{k2} = 2.042\times 10^4\,\frac{1}{s^3}$$

$$k_2 := \frac{P_{k2}}{D_{k2}} \qquad k_2 = -2.296235 \qquad \sigma_2 = 27.709\,\frac{rad}{s}$$

$$f_2(t) := k_2\cdot\exp(-\sigma_2\cdot t)$$



$$k_3 = \left[\frac{(S+\omega_D)\cdot K_x}{D(S)}\right] \Bigg|_{S = -\sigma_3 - j\cdot\omega_3}$$

$$\frac{(S+\sigma_3+j\cdot\omega_3)\cdot K_x}{D(S)} =$$

$$D(S) := (S+\sigma_1)\cdot(S+\sigma_2)\cdot(S+\sigma_3+j\cdot\omega_3)\cdot(S+\sigma_3-j\cdot\omega_3)$$

$$D_{k3}(S) = (S+\sigma_1)\cdot(S+\sigma_2)\cdot(S+\sigma_3+j\omega_3)$$

$$D_{k3} := (-\sigma_3-j\cdot\omega_3+\sigma_1)\cdot(-\sigma_3-j\cdot\omega_3+\sigma_2)\cdot(-\sigma_3-j\cdot\omega_3+\sigma_3-j\cdot\omega_3)$$

$$-\sigma_3 - j\cdot\omega_3 = -35.03 - 17.187i\,\frac{\text{rad}}{\text{s}}$$

$$D_{k3} = 2.37\times 10^4 + 1.899i\times 10^3\,\frac{1}{s^3}$$

$$\text{Re}(D_{k3}) = 2.37\times 10^4\,\frac{1}{s^3} \qquad \text{Im}(D_{k3}) = 1.899\times 10^3\,\frac{1}{s^3}$$

$$P_{k3} := (-\sigma_3+\omega_D-j\cdot\omega_3)\cdot K_x$$

$$P_{k3} = 1.835\times 10^4 - 4.854i\times 10^3\,\frac{1}{s^3}$$

$$\frac{P_{k3}}{D_{k3}} = 0.753 - 0.265i$$

$$R_p := \text{Re}\left(\frac{P_{k3}}{D_{k3}}\right) \qquad I_p := \text{Im}\left(\frac{P_{k3}}{D_{k3}}\right)$$

$$R_p = 0.753 \qquad I_p = -0.265$$

$$k_3 := (R_p + j\cdot I_p)$$

$$k_3 = 0.753 - 0.265i$$

$$\overline{k_3} = 0.753 + 0.265i$$

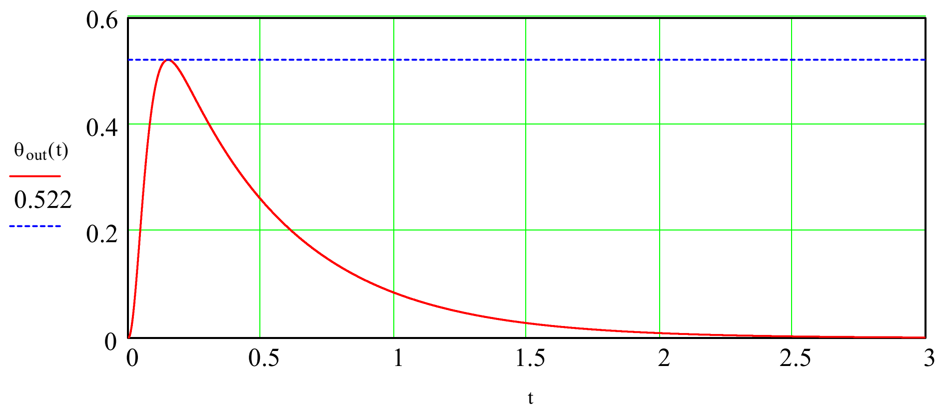
$\gamma := 2 \cdot \mathrm{Re}(k_3)$ $\qquad \delta := 2\,\mathrm{Im}(k_3)$

$\gamma = 1.506$ $\qquad \delta = -0.53$

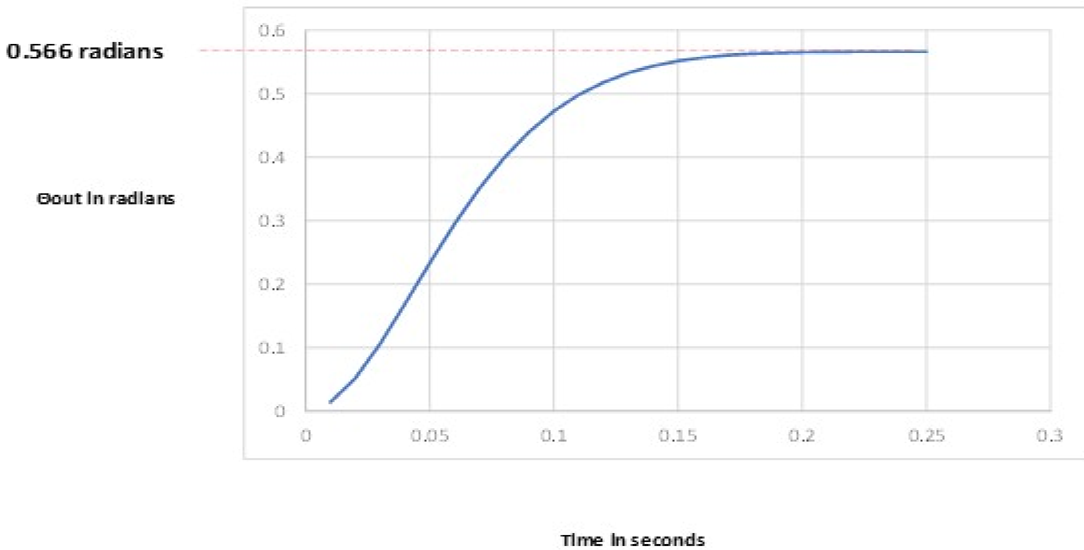
$$f_3(t) := \exp(-\sigma_3 \cdot t) \cdot (\gamma \cdot \cos(\omega_3 \cdot t) + \delta \cdot \sin(\omega_3 \cdot t))$$

$$\theta_{out}(t) := f_1(t) + f_2(t) + f_3(t)$$

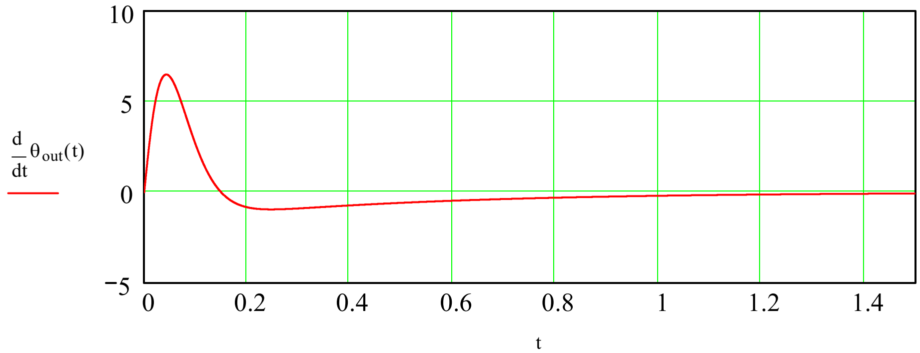
Position Loop Response



Velocity Loop Response for Comparison- Refer to document *Cascade PI Velocity Controler Torque Disturbance Analysis* by Frederick York



Note that the peak Position Output (0.522 rad) is slightly less than that of the Velocity Loop's (0.566 rad) due to the Position Loop begining its control. Of course the position output final vale will be zero due to the servo action.

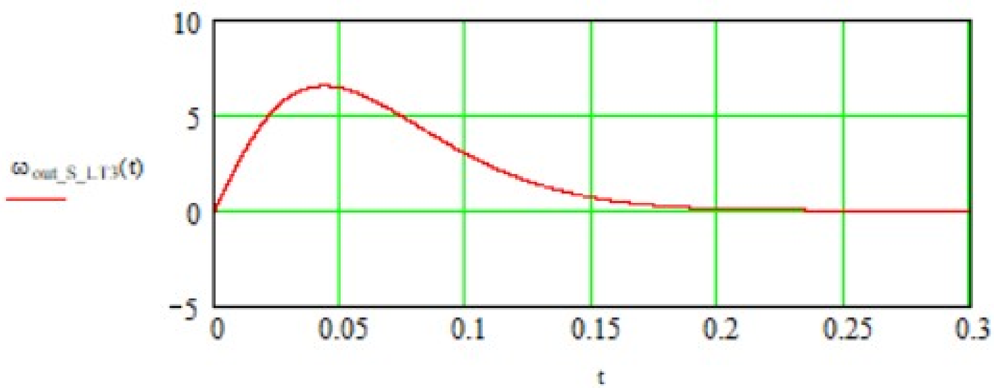


Note the velocity reversal -
to return to the original position.
@ t ~ 0.15 s

Note

$$\int_{0\cdot s}^{0.15\cdot s} \frac{d}{dt}\theta_{out}(t)\, dt = 0.522\,\text{rad}$$

Velocity Loop Response for Comparison



Frequency Domain

Choose

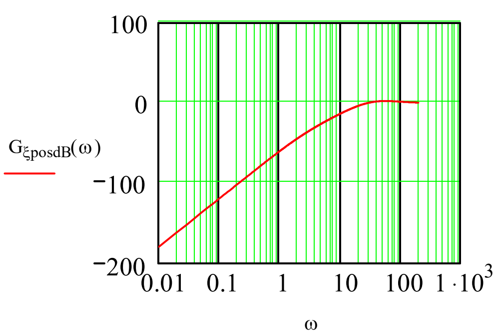
$$T_L(t) := T_{Lp} \cdot \cos(\omega_{test} \cdot t) \qquad \omega_{test} := 1 \cdot \frac{rad}{s}$$

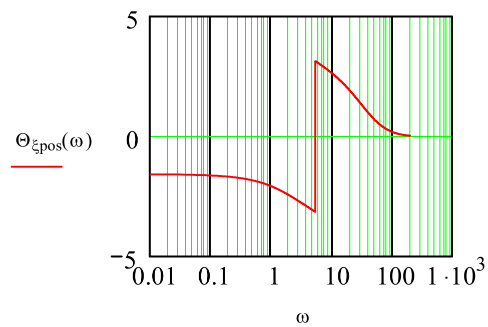
$$t := 0 \cdot s, 0.001 \cdot s .. 60 \cdot s$$

$$S_{\xi pos}(\omega) := \frac{(j \cdot \omega)^3 \cdot (j \cdot \omega + \omega_D)}{(j \cdot \omega)^4 + (j \cdot \omega)^3 \cdot \omega_D + (j \cdot \omega)^2 \cdot \frac{\omega_D^{\,2}}{\chi} + (j \cdot \omega) \cdot \frac{\omega_D^{\,3}}{\chi^3} + K_V \cdot \frac{\omega_D^{\,3}}{\chi^3}}$$

$$S_{\xi pos}(\omega) = \frac{T_\xi(\omega)}{T_L(\omega)}$$

$$G_{\xi posdB}(\omega) := 20 \cdot \log(|S_{\xi pos}(\omega)|) \qquad \Theta_{\xi pos}(\omega) := \arg(S_{\xi pos}(\omega))$$



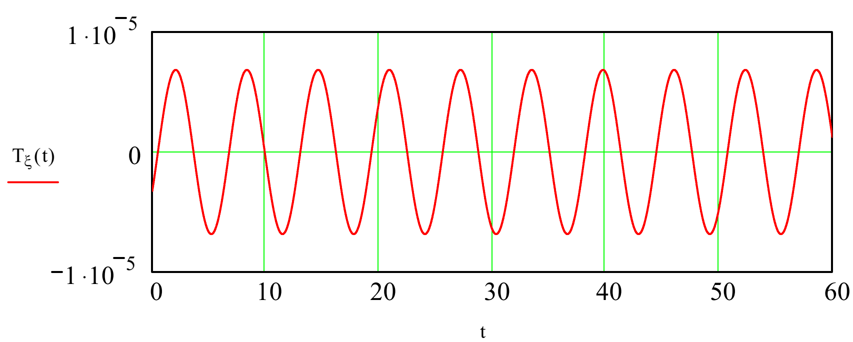


$$G_{\xi posdB}\left(1 \cdot \frac{rad}{s}\right) = -60.275\,dB \qquad \Theta_{\xi pos}\left(1 \cdot \frac{rad}{s}\right) = -118.282\,deg$$

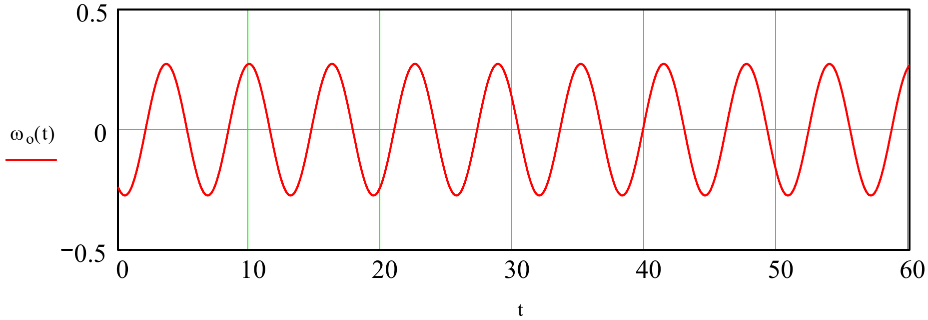
$$T_{Px} := T_{Lp} \cdot |S_{\xi pos}(\omega_{test})| \qquad \Theta := \arg(S_{\xi pos}(\omega_{test}))$$

$$T_{Px} = 6.841 \times 10^{-6}\,N \cdot m \quad P \qquad \Theta = -2.064\,rad \qquad \Theta = -118.282\,deg$$

$$T_\xi(t) := T_{Px} \cdot \cos(\omega_{test} \cdot t + \Theta)$$



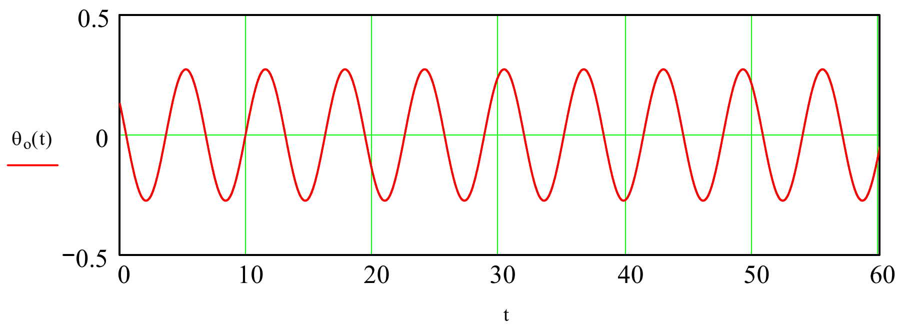
$$\omega_o(t) := \frac{T_{Px}}{J}\cdot\frac{\sin(\omega_{test}\cdot t + \Theta)}{\omega_{test}}$$



$$\theta_o(t) := \frac{-T_{Px}}{J}\cdot\frac{\cos(\omega_{test}\cdot t + \Theta)}{\omega_{test}^{2}}$$

$$\omega_o(9.91839\cdot s) = 0.274\,\frac{rad}{s}$$

$$\omega_o(11.4889\cdot s) = 0\,\frac{rad}{s}$$



$$\theta_o(9.91839\cdot s) = 0\,rad$$

$$\theta_o(11.4889\cdot s) = 0.274\,rad$$

Position Torque Disturbance Reduction

$$TDR(\omega_{test}) := 20\cdot\log\left(\left|\frac{\dfrac{\frac{T_{Px}}{J}\cdot\frac{rad}{s^2}}{\omega_{test}^{2}}}{\dfrac{T_{Lp}}{J}}\right|\right)$$

$$TDR(\omega_{test}) = -60.275\,dB$$

$$G_{\xi posdB}(\omega_{test}) = -60.275\,dB$$

Checks