Alpha-Beta-Gamma Tracking Filter Analysis

The algorithm to estimate the position, velocity and acceleration is:

Predictor, Equations:

$$Y(k+1) = y(k) + T_s \cdot v(k) + \frac{1}{2} T_s^{\,2} \cdot a(k) \qquad \textbf{EQ 1}$$

$$V(k+1) = v(k) + T_s \cdot a(k) \qquad \textbf{EQ 2}$$

$$A(k+1) = a(k) \qquad \textbf{EQ 3}$$

Estimator (Corrector) Equations:

$$y(k) = Y(k) + \alpha \cdot \left(x_o(k) - Y(k)\right) \qquad \textbf{EQ 4}$$

$$v(k) = V(k) + \frac{\beta}{T_s} \cdot \left(x_0(k) - Y(k)\right) \qquad \textbf{EQ 5}$$

$$a(k) = A(k) + \frac{\gamma}{2 \cdot T_s^{\,2}} \cdot \left(x_0(k) - Y(k)\right) \qquad \textbf{EQ 6}$$

$x_0(k)$	Measured Position @ time kT_s
Upper Case	Predictive Parameters (One step ahead)
Lower Case	Estimated Parameters (Smoothing)

T_s = Step Size (samping time)

Alpha-Beta-Gamma Tracking Filter Analysis

To find the Z Transform of the Estimator Equations, Plug EQ 1, 2 and 3 into EQ 4, 5 and 6.
Then rearrange equations so that the input (x_0) is on the right.

Next take the Z transform –

The Z transform of Estimated Parameters

$$(Z+\alpha-1)\cdot y(Z)+(\alpha-1)\cdot T_s\cdot v(Z)+\frac{(\alpha-1)}{2}\cdot {T_s}^2\cdot a(Z)=\alpha\cdot Z\cdot x_0(Z) \qquad \textbf{EQ 7}$$

$$\beta\cdot y(Z)+(Z+\beta-1)\cdot T_s\cdot v(Z)+\left(\frac{\beta}{2}-1\right)\cdot {T_s}^2\cdot a(Z)=\beta\cdot Z\cdot x_0(Z) \qquad \textbf{EQ 8}$$

$$\frac{\gamma}{2}\cdot y(Z)+\frac{\gamma}{2}\cdot T_s\cdot v(Z)+\left(Z+\frac{\gamma}{4}-1\right)\cdot {T_s}^2\cdot a(Z)=\frac{\gamma}{2}\cdot Z\cdot x_0(Z) \qquad \textbf{EQ 9}$$

In Matrix Form

$$\begin{bmatrix} (Z+\alpha-1) & (\alpha-1)\cdot T_s & \left(\frac{\alpha-1}{2}\right)\cdot {T_s}^2 \\ \beta & (Z+\beta-1)\cdot T_s & \left(\frac{\beta}{2}-1\right)\cdot {T_s}^2 \\ \frac{\gamma}{2} & \frac{\gamma}{2}\cdot T_s & \left(Z+\frac{\gamma}{4}-1\right)\cdot {T_s}^2 \end{bmatrix} \begin{pmatrix} y(Z) \\ v(Z) \\ a(Z) \end{pmatrix} = \begin{pmatrix} \alpha\cdot Z\cdot x_0(Z) \\ \beta\cdot Z\cdot x_0(Z) \\ \frac{\gamma}{2}\cdot Z\cdot x_0(Z) \end{pmatrix} \qquad \textbf{EQ 10}$$

Using Cramer's Rule to solve for y(Z), v(Z) and a(Z) results in:

The Determinate, D, is equal to $Ts^3\,\Delta$, where Δ is found to be:

$$\Delta = Z^3+\left(\alpha+\beta+\frac{\gamma}{4}-3\right)\cdot Z^2+\left(-2\cdot\alpha-\beta+\frac{\gamma}{4}+3\right)\cdot Z+(\alpha-1) \qquad \textbf{EQ 11}$$

$$Z\left(\frac{y(k)}{x_0(k)}\right) = \frac{Z\cdot\left[\alpha\cdot Z^2 + \left(-2\cdot\alpha + \beta + \frac{\gamma}{4}\right)\cdot Z + \left(\alpha - \beta + \frac{\gamma}{4}\right)\right]}{\Delta} \qquad \textbf{EQ 12}$$

$$Z\left(\frac{v(k)}{x_0(k)}\right) = \frac{Z\cdot\left[\beta\cdot Z^2 + \left(-2\cdot\beta + \frac{\gamma}{2}\right)\cdot Z + \left(\beta - \frac{\gamma}{2}\right)\right]}{T_s\cdot\Delta} \qquad \textbf{EQ 13}$$

$$Z\left(\frac{a(k)}{x_0(k)}\right) = \frac{Z\cdot\left(\frac{1}{2}\cdot\gamma\cdot Z^2 - \gamma\cdot Z + \frac{1}{2}\cdot\gamma\right)}{T_s^{\,2}\cdot\Delta} \qquad \textbf{EQ 14}$$

To find the Predicted Parameters Z Transform we note that using EQ 1, 2 and 3:

Z Transform of Predictive Parameters:

$$\frac{Y(Z)}{x_0(Z)} = \frac{y(Z)}{Z} + T_s\cdot\frac{v(Z)}{Z} + T_s^{\,2}\cdot\frac{a(Z)}{2\cdot Z} \qquad \textbf{EQ 15}$$

$$\frac{V(Z)}{x_0(Z)} = \frac{v(Z)}{Z} + T_s\cdot\frac{a(Z)}{Z} \qquad \textbf{EQ 16}$$

$$\frac{A(Z)}{x_0(Z)} = \frac{a(Z)}{Z} \qquad \textbf{EQ 17}$$

Which when substituting EQ 12, 13 and 14 into EQ 15, 16 and 17 the following is obtained:

The Z Transform of Predictive Parameters

$$Z\left(\frac{Y(k)}{x_0(k)}\right) = \frac{\left(\alpha + \beta + \frac{\gamma}{4}\right)\cdot Z^2 + \left(-2\cdot\alpha - \beta + \frac{\gamma}{4}\right)\cdot Z + \alpha}{\Delta} \qquad \textbf{EQ 18}$$

$$Z\left(\frac{V(k)}{x_0(k)}\right) = \frac{\left(\beta + \frac{\gamma}{2}\right)\cdot Z^2 + \left(-2\cdot\beta - \frac{\gamma}{2}\right)\cdot Z + \beta}{T_s\cdot\Delta} \qquad \textbf{EQ 19}$$

$$Z\left(\frac{A(k)}{x_0(k)}\right) = \frac{\frac{\gamma}{2}\cdot Z^2 - \gamma\cdot Z + \frac{\gamma}{2}}{{T_s}^2\cdot\Delta} \qquad \textbf{EQ 20}$$

Discrete Time Domain.

<u>Finding the discrete time sequence</u>

for EQ 12 -

$$Z\left(\frac{y(k)}{x_0(k)}\right) = \frac{Z\cdot\left[\alpha\cdot Z^2 + \left(-2\cdot\alpha + \beta + \frac{\gamma}{4}\right)\cdot Z + \left(\alpha - \beta + \frac{\gamma}{4}\right)\right]}{\Delta} \qquad \textbf{EQ 12}$$

Rearrange by factoring the bracketed numerator by Z^2 and the denominator by Z^3 (which of course cancel) leaving-

$$= \frac{\alpha + \left(-2\cdot\alpha + \beta + \frac{\gamma}{4}\right)\cdot Z^{-1} + \left(\alpha - \beta + \frac{\gamma}{4}\right)\cdot Z^{-2}}{1 + \left(\alpha + \beta + \frac{\gamma}{4} - 3\right)\cdot Z^{-1} + \left(-2\cdot\alpha - \beta + \frac{\gamma}{4} + 3\right)\cdot Z^{-2} + (\alpha - 1)\cdot Z^{-3}} \qquad \textbf{EQ 21}$$

Using the relationship $Z(p(k-n)) = Z^{-n}\ p(Z)$

$$y_k = \alpha\cdot x_0(k) + \left(-2\cdot\alpha + \beta + \frac{\gamma}{4}\right)\cdot x_0(k-1) + \left(\alpha - \beta + \frac{\gamma}{4}\right)\cdot x_0(k-2)$$

$$-\left(\alpha + \beta + \frac{\gamma}{4} - 3\right)\cdot y_{k-1} - \left(-2\cdot\alpha - \beta + \frac{\gamma}{4} + 3\right)\cdot y_{k-2} - (\alpha - 1)\cdot y_{k-3} \qquad \textbf{EQ 22}$$

Note that since Mathcad software starts matrix and vector references at 0, k is increased by +3.

This yields-

$$y_{k+3} = \alpha\cdot x_0(k+3) + \left(-2\cdot\alpha + \beta + \frac{\gamma}{4}\right)\cdot x_0(k+2) + \left(\alpha - \beta + \frac{\gamma}{4}\right)\cdot x_0(k+1)$$

$$-\left(\alpha + \beta + \frac{\gamma}{4} - 3\right)\cdot y_{k+2} - \left(-2\cdot\alpha - \beta + \frac{\gamma}{4} + 3\right)\cdot y_{k+1} - (\alpha - 1)\cdot y_k \qquad \textbf{EQ 23}$$

Likewise for EQ 18

$$Z\left(\frac{Y(k)}{x_0(k)}\right) = \frac{\left(\alpha + \beta + \frac{\gamma}{4}\right)\cdot Z^2 + \left(-2\cdot\alpha - \beta + \frac{\gamma}{4}\right)\cdot Z + \alpha}{\Delta} \qquad \textbf{EQ 18}$$

Rearrange by factoring the numerator by Z^2 and the denominator by Z^3 resulting in:

$$= \frac{\left(\alpha + \beta + \frac{\gamma}{4}\right) + \left(-2\cdot\alpha - \beta + \frac{\gamma}{4}\right)\cdot Z^{-1} + \alpha\cdot Z^{-2}}{Z\cdot\left[1 + \left(\alpha + \beta + \frac{\gamma}{4} - 3\right)\cdot Z^{-1} + \left(-2\cdot\alpha - \beta + \frac{\gamma}{4} + 3\right)\cdot Z^{-2} + (\alpha - 1)\cdot Z^{-3}\right]} \qquad \textbf{EQ 24}$$

Using the relationships

$Z(Y(k+1)) = Z\cdot Y(Z) - Z\cdot Y(0)$ and $Z\left(x_0(k-n)\right) = Z^{-n}\cdot x_0(Z)$

With $Y(0) = 0$

Again, k is increased (for Mathcad) but this time by 2 which results in

$$Y_{k+3} = \left(\alpha + \beta + \frac{\gamma}{4}\right)\cdot x_0(k+2) + \left(-2\cdot\alpha - \beta + \frac{\gamma}{4}\right)\cdot x_0(k+1) + \alpha\cdot x_0(k)$$

$$-\left(\alpha + \beta + \frac{\gamma}{4} - 3\right)\cdot Y_{k+2} - \left(-2\cdot\alpha - \beta + \frac{\gamma}{4} + 3\right)\cdot Y_{k+1} - (\alpha - 1)\cdot Y_k \qquad \textbf{EQ 25}$$

The remaining discrete equations are given below:

$$v_{k+3} = \frac{\beta \cdot x_0(k+3) + \left(-2\cdot\beta + \frac{\gamma}{2}\right)\cdot x_0(k+2) + \left(\beta - \frac{\gamma}{2}\right)\cdot x_0(k+1)}{T_s}$$

$$\frac{-\left(\alpha + \beta + \frac{\gamma}{4} - 3\right)\cdot v_{k+2} - \left(-2\cdot\alpha - \beta + \frac{\gamma}{4} + 3\right)\cdot v_{k+1} - (\alpha - 1)\cdot v_k}{T_s} \qquad \textbf{EQ 26}$$

$$V_{k+3} = \frac{\left(\beta + \frac{\gamma}{2}\right)\cdot x_0(k+2) + \left(-2\cdot\beta - \frac{\gamma}{2}\right)\cdot x_0(k+1) + \beta\cdot x_0(k)}{T_s}$$

$$\frac{-\left(\alpha + \beta + \frac{\gamma}{4} - 3\right)\cdot V_{k+2} - \left(-2\cdot\alpha - \beta + \frac{\gamma}{4} + 3\right)\cdot V_{k+1} - (\alpha - 1)\cdot V_k}{T_s} \qquad \textbf{EQ 27}$$

$$a_{k+3} = \frac{\frac{\gamma}{2}\cdot x_0(k+3) - \gamma\cdot x_0(k+2) + \frac{\gamma}{2}\cdot x_0(k+1)}{T_s^{\,2}}$$

$$\frac{-\left(\alpha + \beta + \frac{\gamma}{4} - 3\right)\cdot a_{k+2} - \left(-2\cdot\alpha - \beta + \frac{\gamma}{4} + 3\right)\cdot a_{k+1} - (\alpha - 1)\cdot a_k}{T_s^{\,2}} \qquad \textbf{EQ 28}$$

$$A_{k+3} = \frac{\frac{\gamma}{2}\cdot x_0(k+2) - \gamma\cdot x_0(k+1) + \frac{\gamma}{2}\cdot x_0(k)}{T_s^{\,2}}$$

$$\frac{-\left(\alpha + \beta + \frac{\gamma}{4} - 3\right)\cdot A_{k+2} - \left(-2\cdot\alpha - \beta + \frac{\gamma}{4} + 3\right)\cdot A_{k+1} - (\alpha - 1)\cdot A_k}{T_s^{\,2}} \qquad \textbf{EQ 29}$$

Frequency Domain.

To find the frequency response for the transfer functions given in EQ 12, 13, 14 and EQ 18, 19 and 20 substitute Z = exp(+jωTs). The ωTs range is from 0 *rad to π rad (Nyquist Rate).

We will now compare EQ 12 and EQ 18 in the frequency domain.

$$Z\left(\frac{y(k)}{x_0(k)}\right) = G_y(Z) \qquad \textbf{EQ 30}$$

With $Z(\omega) := \exp(j\cdot\omega\cdot T_s)$

$$G_y(\omega) := \frac{Z(\omega)\cdot\left[\alpha\cdot(Z(\omega)-1)^2 + \left(\beta + \frac{\gamma}{4}\right)\cdot(Z(\omega)-1) + \frac{\gamma}{2}\right]}{Z(\omega)^3 + \left(\alpha + \beta + \frac{\gamma}{4} - 3\right)\cdot Z(\omega)^2 + \left(-2\cdot\alpha - \beta + \frac{\gamma}{4} + 3\right)\cdot Z(\omega) + (\alpha - 1)} \qquad \textbf{EQ 31}$$

$$G_{ydB}(\omega) := 20\cdot\log\left(\left|G_y(\omega)\right|\right) \qquad \textbf{EQ 32} \qquad \textbf{Gain Response}$$

$$\phi_y(\omega) := \arg\left(G_y(\omega)\right) \qquad \textbf{EQ 33} \qquad \textbf{Phase Response}$$

$$Z\left(\frac{Y(k)}{x_0(k)}\right) = G_Y(Z) \qquad \textbf{EQ 34}$$

With $Z(\omega) := \exp(j\cdot\omega\cdot T_s)$

$$G_Y(\omega) := \frac{\left(\alpha + \beta + \frac{\gamma}{4}\right)\cdot(Z(\omega))^2 + \left(-2\cdot\alpha - \beta + \frac{\gamma}{4}\right)\cdot Z(\omega) + \alpha}{Z(\omega)^3 + \left(\alpha + \beta + \frac{\gamma}{4} - 3\right)\cdot Z(\omega)^2 + \left(-2\cdot\alpha - \beta + \frac{\gamma}{4} + 3\right)\cdot Z(\omega) + (\alpha - 1)} \qquad \textbf{EQ 35}$$

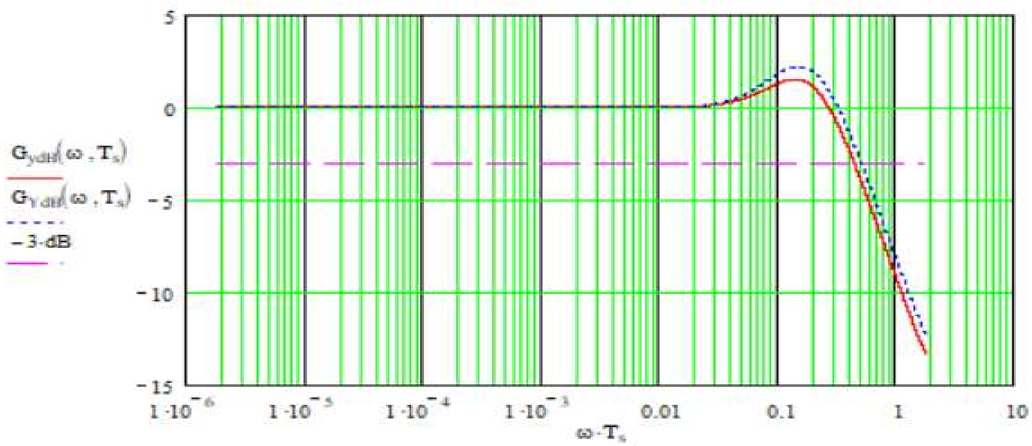
$$G_{YdB}(\omega) := 20\cdot\log\left(\left|G_Y(\omega)\right|\right) \qquad \textbf{EQ 36}$$

$$\phi_Y(\omega) := \arg\left(G_Y(\omega)\right) \qquad \textbf{EQ 37}$$

Example 1

Let the following parameters be set as follows:
Note that throughout we don't specify Ts. When we plot a function vs ωTs we obtain a consistent view, as scaling remains the same for any Ts.

Gain Comparison –



Phase Comparison

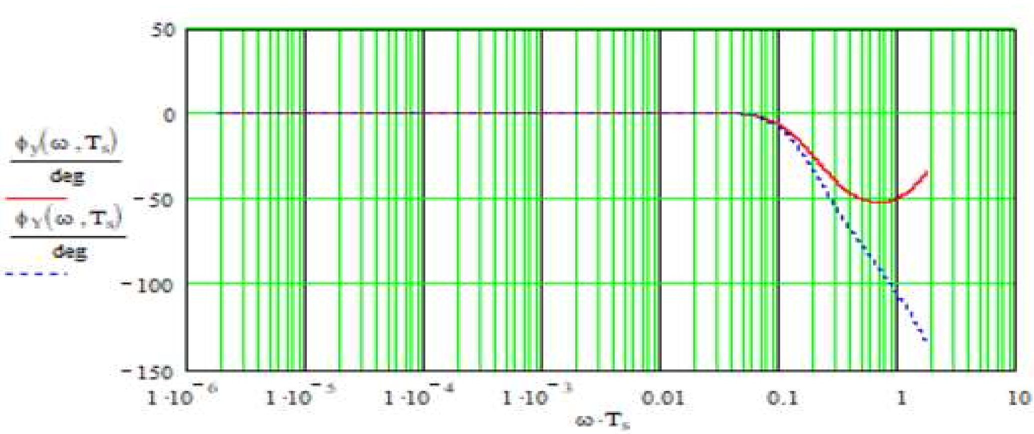


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Figure 1A and 1B
Gain and Phase Comparisons

Note that the estimated function offers a narrower (by ~ 13%) bandwidth with less gain peaking (~ 0.706 dB) then the predicted gain. The phase response is flattened (improved phase margin, by ~ 13 deg, when used in a closed loop) for the estimated function as well.

Analytical Expressions – From Trend-Following Filters – Part 2/2

By Henry Stern|January 21st, 2021|Research Insights, Factor Investing, Trend Following

Given α

Discounted least squares error (equivalent to Critical Damping)

$$\beta = 3 - 1.5\cdot(1-\alpha)^{\frac{1}{3}} - 1.5\cdot(1-\alpha)^{\frac{2}{3}} - 1.5\cdot\alpha \qquad \textbf{EQ 38}$$

$$\gamma = -3\cdot(1-\alpha)^{\frac{1}{3}} + 3\cdot(1-\alpha)^{\frac{2}{3}} + \alpha \qquad \textbf{EQ 39}$$

Random Velocity

$$\beta = \frac{\alpha^2}{2-\alpha} \qquad \textbf{EQ 40} \qquad\qquad \gamma = \frac{\beta^2}{2\cdot\alpha} \qquad \textbf{EQ 41}$$

Random Acceleration

$$\beta = 4 - 2\cdot\alpha - 4\cdot\sqrt{1-\alpha} \qquad \textbf{EQ 42} \qquad\qquad \gamma = \frac{\beta^2}{2\cdot\alpha} \qquad \text{same as } \textbf{EQ 41}$$

Note that with $\alpha = 0.3$ and using EQ 38 and EQ 39

$\beta = 0.0356$ and $\Upsilon = 1.4085 \times 10^{-3}$.

These were the values used in Example 1.

Analytical Expressions – From Technical Report STEWS-ID-78-3 (1978)
Deals exclusively with estimated parameters.

$$\left|\frac{x_s(k)}{x_0(k)}\right| \text{ min gain @ } \omega\,Ts = \pi \qquad \frac{2\cdot\alpha - \beta}{4 - 2\cdot\alpha - \beta} \qquad \textbf{EQ 43}$$

$$\left|\frac{x_s(k)}{x_0(k)}\right| \text{ max} \quad @ \quad (\omega\cdot T_s)\text{ max} = \sqrt{\frac{\beta}{2}} \qquad \textbf{EQ 44}$$

$$\left|\frac{x_s(k)}{x_0(k)}\right| = 1 \quad \text{when} \quad \omega\cdot T_s = \operatorname{acos}(1-\beta) \sim \sqrt{2\cdot\beta} \qquad \textbf{EQ 45}$$

$$\left|\frac{x_s(k)}{x_0(k)}\right| = \frac{1}{\sqrt{2}} \quad \text{when} \quad \omega\cdot T_s = \sqrt{2\cdot\beta}\cdot 10^{-0.15\cdot\frac{\ln\left(\frac{\pi}{\sqrt{2\cdot\beta}}\right)}{\ln\left(\frac{\alpha}{2}\right)}} \qquad \textbf{EQ 46}$$

$$\left|\frac{x_s(k)}{x_0(k)}\right| = \frac{1}{2} \quad \text{when} \quad \omega\cdot T_s = \sqrt{2\cdot\beta}\cdot 10^{-0.3\cdot\frac{\ln\left(\frac{\pi}{\sqrt{2\cdot\beta}}\right)}{\ln\left(\frac{\alpha}{2}\right)}} \qquad \textbf{EQ 47}$$

For Example 1 and other Parameters:

α	β	ϒ	Item	Description	Exact	Approximate	Notes
0.3	**0.0356**	**1.41*10^-3**	1	Min Gain @ π	-15.5063 dB	-15.5064 dB	
			2	Max Gain ωTs	0.133417 rad	0.133417 rad	1.4938 dB
			3	ωTs for Gain = 0 dB	0.267631 rad	0.267631 rad	
			4	ωTs for Gain = -3 dB	0.440 rad	0.418 rad	-2.644 dB for Approx
			5	ωTs for Gain = -6 dB	0.6584 rad	0.6549 rad	-5.9589 dB for Approx

α	β	ϒ	Item	Description	Exact	Approximate	Notes
0.5	**0.1145**	**8.78*10^-3**	1	Min Gain @ π	-10.26072 dB	-10.26071 dB	
			2	Max Gain ωTs	0.262 rad	0.23928 rad	1.2309 dB
			3	ωTs for Gain = 0 dB	0.4827 rad	0.482349	
			4	ωTs for Gain = -3 dB	0.8244 rad	0.76479 rad	-2.5101 dB for Approx
			5	ωTs for Gain = -6 dB	1.2773 rad	1.22221 rad	-5.69746 dB for Approx

α	β	ϒ	Item	Description	Exact	Approximate	Notes
0.75	**0.3478**	**0.05067**	1	Min Gain @ π	-5.38188 dB	-5.38188 dB	
			2	Max Gain ωTs	0.485 rad	0.40913 rad	0.79836 dB
			3	ωTs for Gain = 0 dB	0.8471 rad	0.843008	
			4	ωTs for Gain = -3 dB	1.61521 rad	1.31412 rad	-1.91669 dB for Approx
			5	ωTs for Gain = -6 dB	N/A See Min Gain	2.11044 rad	-4.32455 dB for Approx

Table 1

Analytical Approximation vs. Actual Data
as a Function of Critically Damped Parameters

Analytical Expressions – From Trend-Following Filters – Part 2/2

By Henry Stern|January 21st, 2021|Research Insights, Factor Investing, Trend Following

Discrete Time – Approximate Overshoot for estimated position function at ωTs $_{maxOS}$

$$n := \frac{\alpha \cdot \beta}{2} - \gamma + \frac{3}{8} \cdot \beta^2 \qquad \textbf{EQ 48}$$

$$M := \frac{3}{8} \cdot \beta^2 \cdot (1 - \alpha) \cdot \left(n + \frac{\beta}{3} \right) \qquad \textbf{EQ 49}$$

$$N := \sqrt{\frac{\beta^3 \cdot (1 - \alpha)^2 \cdot n^2}{16}} \qquad \textbf{EQ 50}$$

$$D := n^2 + \frac{\beta^3}{8} \qquad \textbf{EQ 51}$$

$$\text{Overshoot} := \frac{M}{D} + \frac{M^2 + N^2}{2 \cdot D^2} \qquad \textbf{EQ 52}$$

For Example 1, EQ 52 yields 0.261 while the actual overshoot for y_k is 0.155 ($k = 10$, $k+3 = 13$). Refer to Figure 2 below:

Input is

$$\text{STEP}(k) := 2.5 \cdot \Phi(k)$$

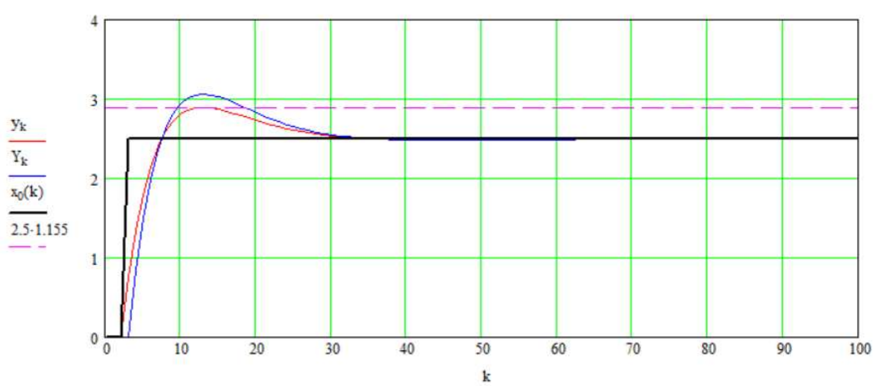

Figure 2
Step Response with Without Noise

For noise performance, refer to the document, *Comparison of Smoothing Estimation Filters for Servo Motor Applications.* This document includes the Alpha-Beta-Gamma Filter noise performance by section (position, velocity and acceleration).

References:

[1] J. E. Steelman (1978**),** *Frequency Response of Alpha-Beta-Gamma Trackers*
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Section 5. Alpha-Beta-Gamma (α−β−γ) Tracking Filter.

[4] Dirk Tenne and Tarunraj Singh (2001), *Optimal Design of α-β-(ϒ) Filters,* Center for Multisource Information Fusion, State University of New York at Buffalo, Buffalo, NY 14269, Pages 1-5.

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